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# The Biequivalence of Locally Cartesian Closed Categories and Martin-Löf Type Theories

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Seely's paper *Locally cartesian closed categories and type theory* contains a well-known result in categorical type theory: that the category of locally cartesian closed categories is equivalent to the category of Martin-Löf type theories with $\Pi, \Sigma$, and extensional identity types. However, Seely's proof relies on the problematic assumption that substitution in types can be interpreted by pullbacks. Here we prove a corrected version of Seely's theorem: that the Bénabou-Hofmann interpretation of Martin-Löf type theory in locally cartesian closed categories yields a biequivalence of 2-categories. To facilitate the technical development we employ categories with families as a substitute for syntactic Martin-Löf type theories. As a second result we prove that if we remove $\Pi$-types the resulting categories with families with only $\Sigma$ and extensional identity types are biequivalent to left exact categories.

## 1. Introduction

It is "well-known" that locally cartesian closed categories (lcccs) are equivalent to Martin-Löf's intuitionistic type theory (Martin-Löf, 1982; Martin-Löf, 1984). But is this result actually *known*? The original proof of (Seely, 1984) contains a flaw, and although the papers by (Curien, 1993) and (Hofmann, 1994) address this flaw, they only show that Martin-Löf type theory can be interpreted in locally cartesian closed categories, but not that this interpretation is an equivalence of categories provided the type theory has $\Pi, \Sigma$, and extensional identity types. Here we complete the work and fully rectify Seely's

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result except that we do not prove an equivalence of categories but a *biequivalence* of 2-categories. In fact, a significant part of the endeavour has been to find an appropriate formulation of the result, and in particular to find a suitable notion analogous to Seely's "interpretation of Martin-Löf theories".

### 1.1. *The coherence problem.*

Seely interpreted substitution in types in Martin-Löf theories by pullbacks in lcccs. For each type substitution we thus need a choice of pullbacks, and this determines the interpretation of the type constructors in the lccc. However, type substitution is also defined syntactically by induction on the type structure, and thus fixed by this interpretation, and there is no reason why the result of this syntactic operation should coincide strictly with the initial choice of pullbacks.

In the paper *Substitution up to isomorphism* (Curien, 1993) described the fundamental nature of this problem. He set out

> ... to solve a difficulty arising from a mismatch between syntax and semantics: in locally cartesian closed categories, substitution is modelled by pullbacks (more generally pseudo-functors), that is, only up to isomorphism, unless split fibrational hypotheses are imposed. ... but not all semantics do satisfy them, and in particular not the general description of the interpretation in an arbitrary locally cartesian closed category. In the general case, we have to show that the isomorphisms between types arising from substitution are *coherent* in a sense familiar to category theorists.

To solve the problem Curien introduced a new calculus with explicit substitutions for Martin-Löf type theory. This calculus has special terms which witness the application of the type equality rule, and these equality witnesses are interpreted as isomorphisms in lcccs. The remaining coherence problem is to show that Curien's calculus is equivalent to the usual formulation of Martin-Löf type theory. Curien proved this result by cut-elimination. It is worth-while noting that this result was proved for a calculus with $\Pi$-types and $\Sigma$-types, but without (extensional) identity types.

Somewhat later, (Hofmann, 1994) gave an alternative solution based on a technique which had been used by (Bénabou, 1985) for constructing a *split* fibration from an arbitrary fibration. In this way Hofmann constructed a model of Martin-Löf type theory with $\Pi$-types, $\Sigma$-types, and extensional identity types from an lccc. Hofmann used Cartmell's categories with attributes (cwas) as his notion of model. This is a split notion of model of Martin-Löf type theory, hence the relevance of Bénabou's construction. However, Seely wanted to prove an equivalence of categories, and (Hofmann, 1994) conjectured:

> We have now constructed a cwa over $\mathcal{C}$ which can be shown to be equivalent to $\mathcal{C}$ in some suitable 2-categorical sense.

Here we spell out and prove this result, and thus fully rectify Seely's theorem. It should be apparent from what follows that this is not a trivial exercise. In our setting the result is a biequivalence analogous to Bénabou's (much simpler) result: that the 2-category of fibrations (with non-strict morphisms) is biequivalent to the 2-category of split fibrations (with non-strict morphisms).

### 1.2. *Our reformulation*

As already mentioned, we have not only corrected but also reformulated Seely's result. We shall now motivate why our reformulation is an improvement while simultaneously the corrected 2-dimensional version of the result. There are three main differences and we discuss each one in turn.

*Categories with families instead of syntactic Martin-Löf theories.* Seely's aim was to relate "syntax" to categorical "semantics", but what is Martin-Löf type theory "syntactically"? If we compare the syntax and inference rules used by Martin-Löf and other authors, we note that different versions usually differ from each other in detail. Moreover, they are usually not completely presented. To remedy this situation (Martin-Löf, 1992) introduced an explicit substitution calculus (see also (Tasistro, 1993)) with the objective of providing a complete and rigorous definition of intuitionistic type theory, and in particular a rigorous treatment of substitution. We shall not employ this substitution calculus directly, but instead the closely related *categories with families (cwfs)* (Dybjer, 1996). Cwfs are models of a variable-free variant of Martin-Löf's substitution calculus which can be defined using category-theoretic terminology. A cwf is a pair $(\mathbb{C}, T)$ where $\mathbb{C}$ is the category of contexts and substitutions, and $T : \mathbb{C}^{op} \to \mathbf{Fam}$ is a functor. An object $\Gamma$ of $\mathbb{C}$ represents a context $x_1 : A_1, \ldots, x_n : A_n$ and an arrow $\gamma : \Delta \to \Gamma$ represents a substitution $x_1 = a_1, \ldots, x_n = a_n$, where $a_1 : A_1, \ldots, a_n : A_n$ are terms in the context $\Delta$. The object part $T(\Gamma)$ represents the family of sets of terms $\{a \mid \Gamma \vdash a : A\}_A$ indexed by types $A$ in context $\Gamma$, and the arrow part $T(\gamma)$ performs the substitution of $\gamma$ in types and terms. The full definition will be given below where we also discuss the equivalence with syntactic Martin-Löf theories.

Cwfs are closely related to Cartmell's cwas and the essentially equivalent *type-categories* in (Pitts, 2000). Cwfs arise by reformulating the axioms of cwas so that they can be read as a syntactic substitution calculus similar to Martin-Löf's. The direct connection with syntax is the reason why we prefer cwfs to cwas or other categorical notions of model of dependent types.

*Pseudo cwf-morphisms instead of Seely's interpretations of Martin-Löf theories.* One advantage of our approach compared to Seely's is that we get a natural definition of morphism of cwfs that preserves the structure of cwfs up to isomorphism. In contrast Seely's notion of "interpretation of Martin-Löf theories" is defined indirectly via the construction of an lccc associated with a Martin-Löf theory, and basically amounts to a functor preserving structure between the corresponding lcccs, rather than directly as something that preserves all the "structure" of Martin-Löf theories.

*Democratic cwfs instead of Seely's categories of closed types.* Seely turned a given Martin-Löf theory into a category where the objects are *closed* types and the morphisms from type $A$ to type $B$ are closed terms of type $A \to B$. Such categories are the objects of Seely's "category of Martin-Löf theories".

Cwfs on the other hand model *open* types and terms. However, to prove our biequivalences we need the additional requirement that they are *democratic*. This means that each context is *represented* by a type. To build local cartesian closed structure in the category of contexts we use available constructions on types and terms, and by democracy such constructions can be moved back and forth between types and contexts. Since Seely worked with closed types only he had no need for democracy.

While carrying out the proof we noticed that if we remove $\Pi$-types the resulting 2-category of cwfs with only $\Sigma$ and extensional identity types is biequivalent to the 2-category of left exact (or finitely complete) categories. We present this result in parallel with the main result. This result would not have been possible in Seely's setting where morphisms are functions of type $A \to B$ and hence $\Pi$-types cannot be removed.

### 1.3. *Plan of the paper.*

An equivalence of categories consists of a pair of functors which are inverses up to natural isomorphism. Biequivalence is the appropriate notion of equivalence for bicategories (Leinster, 1999). Instead of functors we have *pseudofunctors* which only preserve identity and composition up to isomorphism. Instead of natural isomorphisms we have *pseudonatural transformations* which are inverses up to *invertible modification.*

A 2-category is a strict bicategory, and the remainder of the paper consists of constructing two biequivalences of 2-categories. In Section 2 we briefly introduce the standard version of extensional Martin-Löf type theory where substitution is a meta-operation. We also show a version with explicit substitutions. We then define cwfs and explain how they arise as models of a variant of the explicit substitution calculus. We also show how to turn a cwf into an indexed category. In Section 3 we define the 2-categories $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ of democratic cwfs which support extensional identity types and $\Sigma$-types and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ of democratic cwfs which also support $\Pi$-types. We also define the notions of pseudo cwf-morphism and pseudo cwf-transformation. In Section 4 we define the 2-categories $\mathbf{FL}$ of left exact categories and $\mathbf{LCC}$ of lcccs. We show that there are forgetful 2-functors $U : \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}} \to \mathbf{FL}$ and $U : \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}} \to \mathbf{LCC}$. In section 5 we construct the pseudofunctors $H : \mathbf{FL} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ and $H : \mathbf{LCC} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ based on the Bénabou-Hofmann construction. In section 6 we prove that $H$ and $U$ give rise to the biequivalences of $\mathbf{FL}$ and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ and of $\mathbf{LCC}$ and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$. Section 7 concludes. Finally, minor lemmas from sections 3, 4, 5 and 6 have been relegated to an appendix.

At the end of the paper, the reader will find a complete index of notations.

## 2. Martin-Löf Type Theory and Categories with Families

In this section we introduce Martin-Löf type theory with extensional identity types, $\Sigma$-types, and $\Pi$-types. We first present the ordinary version where substitution is a meta-operation and then a *substitution calculus* (Martin-Löf, 1992) where substitution is explicit, that is, it is a syntactic construct. A slight modification of this substitution calculus leads to the *cwf-calculus*, the models of which are cwfs which support extensional identity types, $\Sigma$-types and $\Pi$-types.

We explain why the cwf-calculus is equivalent to the substitution calculus and to the standard presentation of Martin-Löf type theory. Thus cwfs which support extensional identity types, Σ- and Π-types is an appropriate substitute for Seely's Martin-Löf theories.

*Comparison with the equivalence of simply typed lambda calculus and cccs.* In this paper we relate extensional Martin-Löf type theory and lcccs using an explicit substitution calculus and cwfs as stepping stones. It may be worth-while pointing out that also the correspondence between the simply typed lambda calculus and cccs can be explained using similar stepping stones.

To this end we note that if the set of types in context Γ does not depend on Γ, then Π-types degenerate to →-types, and cwfs which support Π-types are precisely models of the simply typed lambda calculus. Such *simply typed cwfs* can be used as a similar stepping stone when explaining the correspondence between the simply typed lambda calculus and cccs, as cwfs when explaining the correspondence between Martin-Löf type theory and lcccs. The following table summarizes the situation.

| | |
|---|---|
| simply typed lambda calculus | Martin-Löf type theory with $\mathrm{I}_{\mathrm{ext}}$, Σ and Π |
| $\lambda\sigma$-calculus (Abadi et al., 1990) | Martin-Löf's substitution calculus with $\mathrm{I}_{\mathrm{ext}}$, Σ and Π |
| simply typed cwfs | cwfs which support $\mathrm{I}_{\mathrm{ext}}$, Σ and Π |
| cccs | lcccs |

Note that the two top rows are syntactic calculi and the two bottom rows are categorical "semantic" notions. Cwfs constitute a "sweet spot" in the spectrum between "syntax" and "semantics": on the one hand they are a categorical notion of model and on the other hand they directly give rise to a variable free explicit substitution calculus.

## 2.1. *Extensional Martin-Löf Type Theory*

The extensional version of intuitionistic type theory was introduced by (Martin-Löf, 1982) and further explained in (Martin-Löf, 1984). Here we consider this theory with only three type formers: extensional identity I, disjoint union Σ, and cartesian product Π of a family of types. We also assume an unspecified set of base types.

If we use de Bruijn indices instead of ordinary variables, the grammars for types and terms are respectively

$$\begin{array}{rcl} A & ::= & \mathrm{I}_A(a,a) \mid \Sigma(A,A) \mid \Pi(A,A) \mid X \\ a & ::= & n \mid \mathrm{r}_{A,a} \mid \mathrm{pair}(a,a) \mid \pi_1(a) \mid \pi_2(a) \mid \lambda(a) \mid \mathrm{ap}(a,a) \end{array}$$

Here $X$ is a base type, $n$ is a de Bruijn index, and $\mathrm{r}_{A,a}$ is the canonical proof of the reflexivity of identity $\mathrm{I}_A(a,a)$.

Contexts Γ are here lists of types. We also introduce *substitutions* $\gamma$ as lists of terms:

$$\begin{array}{rcl} \Gamma & ::= & [\,] \mid \Gamma\cdot A \\ \gamma & ::= & \langle\rangle \mid \langle\gamma,a\rangle \end{array}$$

The inference rules for extensional identity types are

I-formation:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash a' : A}{\Gamma \vdash \mathrm{I}_A(a, a') \text{ type}}$$

I-introduction:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{r}_{A,a} : \mathrm{I}_A(a, a)}$$

I-elimination:

$$\frac{\Gamma \vdash c : \mathrm{I}_A(a, a')}{\Gamma \vdash a = a' : A} \qquad \frac{\Gamma \vdash c : \mathrm{I}_A(a, a')}{\Gamma \vdash c = \mathrm{r}_{A,a} : \mathrm{I}_A(a, a')}$$

Note that neither of the above I-elimination rules are valid for *intensional identity types* which instead have elimination and equality rules for the operator J (Martin-Löf, 1975; Martin-Löf, 1986).

We refer to (Martin-Löf, 1984) for the other inference rules of extensional type theory. When formulating these rules we make use of the meta-operation of *substitution*. We can define the operations of simultaneous substitution $A[\gamma]$ and $a[\gamma]$ respectively of a list of terms $\gamma$ for the free variables in a type $A$ or a term $a$ by induction on $A$ and $a$, respectively. Moreover, we can define the composition $\delta \circ \gamma$ of substitutions $\delta$ and $\gamma$ and an identity substitution $\mathrm{id}_\Gamma = \langle \cdots \langle \langle \rangle, n-1 \rangle \cdots , 0 \rangle$, where $n$ is the length of $\Gamma$. Note that all these are meta-operations on syntax.

### 2.2. *Explicit Substitution Calculus for Martin-Löf Type Theory*

An *explicit substitution calculus* for the theory is obtained by instead introducing new type and term constructors for the above mentioned meta-operations. To this end we add the following productions to the above grammars:

$$\begin{array}{rcl} A & ::= & \cdots \mid A[\gamma] \\ a & ::= & \cdots \mid a[\gamma] \\ \gamma & ::= & \cdots \mid \mathrm{id}_\Gamma \mid \gamma \circ \gamma \end{array}$$

There are typing rules for these new constructions and equations which ensure that the explicit substitution calculus is equivalent to the original version of Martin-Löf type theory.

In this way we arrive at a substitution calculus similar to the one in (Martin-Löf, 1992) (see also (Tasistro, 1993)), although Martin-Löf's calculus uses ordinary variables rather than de Bruijn indices and has some other features such as a special judgement for subcontexts.

### 2.3. *Categories with Families*

As mentioned above we shall use categories with families (cwfs) instead of the usual above mentioned syntactic presentation of Martin-Löf type theory. Cwfs can be presented as models of a variable-free generalized algebraic formulation of the most basic rules of dependent type theory (Dybjer, 1996). This *cwf-calculus* can be seen as arising from the explicit substitution calculus above by (i) removing the de Bruijn indices $n$ and (ii) adding constructors for *projections*

$$\begin{array}{rcl} a & ::= & \cdots \mid \mathrm{q}_{\Gamma,A} \\ \gamma & ::= & \cdots \mid \mathrm{p}_{\Gamma,A} \end{array}$$

with appropriate typing rules and equations, such as the projection laws $\mathrm{p}_{\Gamma,A} \circ \langle \gamma, a \rangle = \gamma$ and $\mathrm{q}_{\Gamma,A}[\langle \gamma, a \rangle] = a$. The first projection $\mathrm{p}_{\Gamma,A}$ is the *display map* (Taylor, 1999) for the type $A$ in context $\Gamma$ in categorical semantics. The second projection $\mathrm{q}_{\Gamma,A}$ represents the rule of assumption for the last variable $0 : A$ in the context $\Gamma \cdot A$.

The equivalence of the cwf-calculus and the explicit substitution calculus is obtained in one direction by translating a de Bruijn number $n$ to a term $\mathrm{q}[\mathrm{p}^n]$ and in the other direction by translating $\mathrm{q}_{\Gamma,A}$ to the de Bruijn number 0 and $\mathrm{p}_{\Gamma,A}$ to $\langle \cdots \langle \langle \rangle, n-1 \rangle \cdots, 1 \rangle$, where $n$ is the length of $\Gamma$.

The models of this calculus are cwfs which support extensional identity types, $\Sigma$-, and $\Pi$-types as defined below.

The definition of cwfs can be presented using category-theoretic terminology as follows.

**Definition 1.** Let **Fam** be the category of families of sets defined as follows. An object is a pair $(A, B)$ where $A$ is a set and $B(x)$ is a family of sets indexed by $x \in A$. A morphism with source $(A, B)$ and target $(A', B')$ is a pair consisting of a function $f : A \to A'$ and a family of functions $g(x) : B(x) \to B'(f(x))$ indexed by $x \in A$.

Note that **Fam** is equivalent to the arrow category $\mathbf{Set}^{\to}$.

**Definition 2.** A **category with families (cwf)** $(\mathbb{C}, T)$ consists of the following data:

- A base category $\mathbb{C}$. Its objects represent *contexts* and its morphisms represent *substitutions*. The identity map is denoted by $\mathrm{id}_\Gamma : \Gamma \to \Gamma$ and the composition of maps $\gamma : \Delta \to \Gamma$ and $\delta : \Xi \to \Delta$ is denoted by $\gamma \circ \delta$ or more briefly by $\gamma\delta : \Xi \to \Gamma$.
- A functor $T : \mathbb{C}^{op} \to \mathbf{Fam}$. $T(\Gamma)$ is a pair, where the first component represents the set $\mathrm{Type}(\Gamma)$ of *types* in context $\Gamma$, and the second component represents the type-indexed family $(\Gamma \vdash A)_{A \in \mathrm{Type}(\Gamma)}$ of sets of *terms* in context $\Gamma$. We write $a : \Gamma \vdash A$ for a term $a \in \Gamma \vdash A$. Moreover, if $\gamma$ is a morphism in $\mathbb{C}$, then $T(\gamma)$ is a pair consisting of the *type substitution* function $A \mapsto A[\gamma]$ and the type-indexed family of *term substitution* functions $a \mapsto a[\gamma]$.
- A *terminal object* $[]$ of $\mathbb{C}$ which represents the *empty context* and a terminal map $\langle \rangle_\Delta : \Delta \to []$ which represents the *empty substitution*.
- A *context comprehension* which to an object $\Gamma$ in $\mathbb{C}$ and a type $A \in \mathrm{Type}(\Gamma)$ associates an object $\Gamma \cdot A$ of $\mathbb{C}$, a morphism $\mathrm{p}_{\Gamma,A} : \Gamma \cdot A \to \Gamma$ of $\mathbb{C}$ and a term $\mathrm{q}_{\Gamma,A} \in \Gamma \cdot A \vdash A[\mathrm{p}_{\Gamma,A}]$ such the following universal property holds: for each object $\Delta$ in $\mathbb{C}$, morphism $\gamma : \Delta \to \Gamma$, and term $a \in \Delta \vdash A[\gamma]$, there is a unique morphism $\delta = \langle \gamma, a \rangle : \Delta \to \Gamma \cdot A$, such that $\mathrm{p}_{\Gamma,A} \circ \delta = \gamma$ and $\mathrm{q}_{\Gamma,A}[\delta] = a$. (We remark that a related notion of comprehension for hyperdoctrines was introduced by (Lawvere, 1970).)

We often omit some arguments of operations and write id instead of $\mathrm{id}_\Gamma$, $\mathrm{p}_A$ or p instead of $\mathrm{p}_{\Gamma,A}$, etc. Moreover, we often write composition as juxtaposition: $\gamma\delta = \gamma \circ \delta$.

The details of the interpretation of Martin-Löf type theory in cwfs can be found in (Hofmann, 1996). The equivalence of cwfs and Martin-Löf type theory is proved by (Mimram, 2004).

### 2.4. *Support for Type Formers*

We shall now define what it means that a cwf supports extra structure corresponding to the various type formers of Martin-Löf type theory. These definitions amount to a direct encoding of the inference rules for these type formers in the variable-free language of cwfs.

**Definition 3.** A cwf *supports extensional identity types* provided the following conditions hold:

**Form.** If $A \in \mathrm{Type}(\Gamma)$ and $a, a' : \Gamma \vdash A$, there is $\mathrm{I}_A(a, a') \in \mathrm{Type}(\Gamma)$,
**Intro.** If $a : \Gamma \vdash A$, there is $\mathrm{r}_{A,a} : \Gamma \vdash \mathrm{I}_A(a, a)$,
**Elim.** If $c : \Gamma \vdash \mathrm{I}_A(a, a')$ then $a = a'$ and $c = \mathrm{r}_{A,a}$,

and we also have stability under substitution: if $\gamma : \Delta \to \Gamma$ then

$$\begin{array}{rcl} \mathrm{I}_A(a, a')[\gamma] & = & \mathrm{I}_{A[\gamma]}(a[\gamma], a'[\gamma]) \\ \mathrm{r}_{A,a}[\gamma] & = & \mathrm{r}_{A[\gamma],a[\gamma]} \end{array}$$

**Definition 4.** A cwf *supports* $\Sigma$-*types* iff the following conditions hold:

**Form.** If $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, there is $\Sigma(A, B) \in \mathrm{Type}(\Gamma)$,
**Intro.** If $a : \Gamma \vdash A$ and $b : \Gamma \vdash B[\langle \mathrm{id}, a\rangle]$, there is $\mathrm{pair}(a, b) : \Gamma \vdash \Sigma(A, B)$,
**Elim.** If $c : \Gamma \vdash \Sigma(A, B)$, there are $\pi_1(c) : \Gamma \vdash A$ and $\pi_2(c) : \Gamma \vdash B[\langle \mathrm{id}, \pi_1(c)\rangle]$ such that

$$\begin{array}{rcl} \pi_1(\mathrm{pair}(a, b)) & = & a \\ \pi_2(\mathrm{pair}(a, b)) & = & b \\ \mathrm{pair}(\pi_1(c), \pi_2(c)) & = & c \end{array}$$

and we also have stability under substitution. If $\gamma : \Delta \to \Gamma$ then

$$\begin{array}{rcl} \Sigma(A, B)[\gamma] & = & \Sigma(A[\gamma], B[\langle \gamma \circ \mathrm{p}, \mathrm{q}\rangle]) \\ \mathrm{pair}(a, b)[\gamma] & = & \mathrm{pair}(a[\gamma], b[\gamma]) \\ \pi_1(c)[\gamma] & = & \pi_1(c[\gamma]) \\ \pi_2(c)[\gamma] & = & \pi_2(c[\gamma]) \end{array}$$

Note that in a cwf which supports extensional identity types and $\Sigma$-types surjective pairing, $\mathrm{pair}(\pi_1(c), \pi_2(c)) = c$, follows from the other conditions (Martin-Löf, 1984).

**Definition 5.** A cwf *supports* $\Pi$-*types* iff the following conditions hold:

**Form.** If $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, there is $\Pi(A, B) \in \mathrm{Type}(\Gamma)$.
**Intro.** If $b : \Gamma \cdot A \vdash B$, there is $\lambda(b) : \Gamma \vdash \Pi(A, B)$.

**Elim.** If $c : \Gamma \vdash \Pi(A, B)$ and $a : \Gamma \vdash A$ then there is a term $\mathrm{ap}(c, a) : \Gamma \vdash B[\langle \mathrm{id}, a\rangle]$ such that

$$\begin{aligned}\mathrm{ap}(\lambda(b), a) &= b[\langle \mathrm{id}, a\rangle] \\ \lambda(\mathrm{ap}(c[\mathrm{p}], \mathrm{q})) &= c\end{aligned}$$

and we also have stability under substitution. If $\gamma : \Delta \to \Gamma$ then

$$\begin{aligned}\Pi(A, B)[\gamma] &= \Pi(A[\gamma], B[\langle \gamma \circ \mathrm{p}, \mathrm{q}\rangle]) \\ \lambda(b)[\gamma] &= \lambda(b[\langle \gamma \circ \mathrm{p}, \mathrm{q}\rangle]) \\ \mathrm{ap}(c, a)[\gamma] &= \mathrm{ap}(c[\gamma], a[\gamma])\end{aligned}$$

### 2.5. *Democracy*

Finally, we introduce democracy, a notion which is used in our proof for moving constructions back and forth between contexts and types.

**Definition 6.** A cwf $(\mathbb{C}, T)$ is *democratic* iff for each object $\Gamma$ of $\mathbb{C}$ there is $\overline{\Gamma} \in \mathrm{Type}([])$ and an isomorphism $\Gamma \cong_{\gamma_\Gamma} [] \cdot \overline{\Gamma}$. Each substitution $\delta : \Delta \to \Gamma$ can then be represented by the term $\overline{\delta} = \mathrm{q}[\gamma_\Gamma \delta \gamma_\Delta^{-1}] : [] \cdot \overline{\Delta} \vdash \overline{\Gamma}[\mathrm{p}]$.

Democracy does not correspond to a construction in Martin-Löf type theory. However, a cwf generated inductively by the standard rules of Martin-Löf type theory (or by the rules for the cwf-calculus) is democratic provided it supports a one element type $\mathrm{N}_1$ and $\Sigma$-types, since we can define $\overline{[]} = \mathrm{N}_1$ and $\overline{\Gamma \cdot A} = \Sigma(\overline{\Gamma}, A[\gamma_\Gamma^{-1}])$.

### 2.6. *The Indexed Category of Types in Context*

We shall now define the indexed category associated with a cwf. This will play a crucial role and in particular introduce the notion of *isomorphism* of types.

**Proposition 1 (The Context-Indexed Category of Types).** If $(\mathbb{C}, T)$ is a cwf, then we can define a functor $\boldsymbol{T} : \mathbb{C}^{op} \to \mathbf{Cat}$ as follows:

— The objects of $\boldsymbol{T}(\Gamma)$ are types in $\mathrm{Type}(\Gamma)$. If $A, B \in \mathrm{Type}(\Gamma)$, then a morphism in $\boldsymbol{T}(\Gamma)(A, B)$ is a morphism $f : \Gamma \cdot A \to \Gamma \cdot B$ in $\mathbb{C}$ such that $\mathrm{p}f = \mathrm{p}$.
— If $\gamma : \Delta \to \Gamma$ in $\mathbb{C}$, then $\boldsymbol{T}(\gamma) : \mathrm{Type}(\Gamma) \to \mathrm{Type}(\Delta)$ maps an object $A \in \mathrm{Type}(\Gamma)$ to $A[\gamma]$ and a morphism $\delta : \Gamma{\cdot}A \to \Gamma{\cdot}B$ to $\langle \mathrm{p}_{A[\gamma]}, \mathrm{q}_B[\delta\langle \gamma \circ \mathrm{p}_{A[\gamma]}, \mathrm{q}_{A[\gamma]}\rangle]\rangle : \Delta{\cdot}A[\gamma] \to \Delta{\cdot}B[\gamma]$.

We write $A \cong_f B$ if $f : A \to B$ is an isomorphism in $\boldsymbol{T}(\Gamma)$. If $a : \Gamma \vdash A$, we write $\{f\}(a) = q[f\langle id, a\rangle] : \Gamma \vdash B$ for the *coercion* of $a$ to type $B$ and $a =_f b$ if $a = \{f\}(b)$. Coercions compose, as we can check by the following direct calculation.

$$\begin{aligned}\{f_2\}(\{f_1\}(a)) &= \mathrm{q}[f_2\langle \mathrm{id}, \mathrm{q}[f_1\langle \mathrm{id}, a\rangle]\rangle] \\ &= \mathrm{q}[f_2\langle \mathrm{p}f_1\langle \mathrm{id}, a\rangle, \mathrm{q}[f_1\langle \mathrm{id}, a\rangle]\rangle] \\ &= \mathrm{q}[f_2\langle \mathrm{p}, \mathrm{q}\rangle f_1\langle \mathrm{id}, a\rangle] \\ &= \mathrm{q}[f_2 f_1\langle \mathrm{id}, a\rangle] \\ &= \{f_2 f_1\}(a)\end{aligned}$$

where we use the definition of coercions and manipulation of cwf combinators. Coercions also preserve substitution, in the sense that for all $\delta : \Delta \to \Gamma$, isomorphic types $A \cong_f A'$ in $\mathrm{Type}(\Gamma)$ and term $a : \Gamma \vdash A$ we have $(\{f\}(a))[\delta] = \{\boldsymbol{T}(\delta)(f)\}(a[\delta])$ (see Lemma 7 in the Appendix for details).

Seely's category **ML** of Martin-Löf theories (Seely, 1984) is essentially the category of categories $\boldsymbol{T}([])$ of closed types. We have the following alternative formulation of democracy:

**Proposition 2.** $(\mathbb{C}, T)$ is democratic iff the functor from $\boldsymbol{T}([])$ to $\mathbb{C}$, which maps a closed type $A$ to the context $[] \cdot A$, is an equivalence of categories.

2.7. *Fibres, slices and lcccs.*

Seely's interpretation of type theory in lcccs relies on the idea that a type $A \in \mathrm{Type}(\Gamma)$ can be interpreted as its *display map*, that is, a morphism with codomain $\Gamma$ (corresponding to our $\mathrm{p}_{\Gamma,A}$). For instance, the type $\texttt{list}(n)$ of lists of length $n : \texttt{nat}$ is interpreted as the function $l : \texttt{list} \to \texttt{nat}$ which to each list associates its length. Hence, types and terms in context $\Gamma$ are interpreted in the *slice category* $\mathbb{C}/\Gamma$, since terms are interpreted as global sections. Syntactic types are connected with types-as-display-maps by the following result, an analogue of which was one of the cornerstones of Seely's paper.

**Proposition 3.** If $(\mathbb{C}, T)$ is democratic and supports extensional identity and $\Sigma$-types, then $\boldsymbol{T}(\Gamma)$ and $\mathbb{C}/\Gamma$ are equivalent categories for all $\Gamma$.

*Proof.* To each object (type) $A$ in $\boldsymbol{T}(\Gamma)$ we associate the object $\mathrm{p}_A$ in $\mathbb{C}/\Gamma$. A morphism from $A$ to $B$ in $\boldsymbol{T}(\Gamma)$ is by definition a morphism from $\mathrm{p}_{\Gamma,A}$ to $\mathrm{p}_{\Gamma,B}$ in $\mathbb{C}/\Gamma$.

Conversely, to each object $\delta : \Delta \to \Gamma$ of $\mathbb{C}/\Gamma$ we associate a type in $\mathrm{Type}(\Gamma)$. This is the inverse image $x : \Gamma \vdash \mathrm{Inv}(\delta)(x)$ which is defined type-theoretically by

$$\mathrm{Inv}(\delta)(x) = \Sigma y : \overline{\Delta}.\mathrm{I}_{\overline{\Gamma}}(\overline{x}, \overline{\delta}(y))$$

written in ordinary notation. In cwf combinator notation it becomes

$$\mathrm{Inv}(\delta) = \Sigma(\overline{\Delta}[\langle\rangle], \mathrm{I}_{\overline{\Gamma}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}], \overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) \in \mathrm{Type}(\Gamma)$$

These associations yield an equivalence of categories since $\mathrm{p}_{\mathrm{Inv}(\delta)}$ and $\delta$ are isomorphic in $\mathbb{C}/\Gamma$:

$$\begin{tikzcd} \Gamma \cdot \mathrm{Inv}(\delta) \arrow[rr, bend left, "\xi_\delta"] \arrow[dr, "\mathrm{p}_{\mathrm{Inv}(\delta)}"'] & & \Delta \arrow[ll, bend left, "\xi_\delta^{-1}"] \arrow[dl, "\delta"] \\ & \Gamma & \end{tikzcd}$$

The isomorphism is defined as follows:

$$\begin{aligned} \xi_\delta &= \gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle \\ \xi_\delta^{-1} &= \langle\delta, \mathrm{pair}(\mathrm{q}[\gamma_\Delta], \mathrm{r}_{\overline{\Gamma}[\langle\rangle]})\rangle \end{aligned}$$

It is easy to show that they have the right types, and that $\xi_\delta \xi_\delta^{-1} = \mathrm{id}_\Delta$. For the other equality, we have $\xi_\delta^{-1}\xi_\delta = \langle \delta\gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle, \mathrm{pair}(\pi_1(\mathrm{q}), \mathrm{r}_{\overline{\Gamma[\langle\rangle]}})\rangle$. By the property of extensional identity types $\mathrm{q}[\gamma_\Gamma \mathrm{p}]$ and $\overline{\delta}[\langle\langle\rangle, \pi_1(\mathrm{q})\rangle]$ are equal terms in context $\Gamma \cdot \mathrm{Inv}(\delta)$, so $\gamma_\Gamma^{-1}\langle\langle\rangle, \mathrm{q}[\gamma_\Gamma \mathrm{p}]\rangle = \mathrm{p}$ and $\gamma_\Gamma^{-1}\langle\langle\rangle, \overline{\delta}[\langle\langle\rangle, \pi_1(\mathrm{q})\rangle]\rangle = \delta\gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle$ are equal substitutions. Likewise, $\mathrm{r}_{\overline{\Gamma[\langle\rangle]}} = \pi_2(\mathrm{q})$ by uniqueness of identity proofs, therefore $\xi_\delta^{-1}\xi_\delta = \mathrm{id}_{\Gamma \mathrm{Inv}(\delta)}$. □

It is easy to see that $\boldsymbol{T}(\Gamma)$ has binary products if the cwf supports $\Sigma$-types and exponentials if it supports $\Pi$-types. Simply define $A \times B = \Sigma(A, B[\mathrm{p}])$ and $B^A = \Pi(A, B[\mathrm{p}])$. Hence by Proposition 9 it follows that $\mathbb{C}/\Gamma$ has products and $\mathbb{C}$ has finite limits in any democratic cwf which supports extensional identity types and $\Sigma$-types. If it supports $\Pi$-types too, then $\mathbb{C}/\Gamma$ is cartesian closed and $\mathbb{C}$ is locally cartesian closed.

## 3. The 2-Category of Categories with Families

### 3.1. *Pseudo Cwf-Morphisms*

A notion of *strict cwf-morphism* between cwfs $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ was defined by (Dybjer, 1996). It is a pair $(F, \sigma)$, where $F : \mathbb{C} \to \mathbb{C}'$ is a functor and $\sigma : T \stackrel{\bullet}{\to} T'F$ is a natural transformation of family-valued functors, such that terminal objects and context comprehension are preserved on the nose. Here we need a notion of pseudo cwf-morphism analogous to that of *strict* cwf-morphism, but which only preserves cwf-structure up to coherent isomorphism. The pseudo-natural transformations needed to prove our biequivalences will be families of cwf-morphisms which do not preserve cwf-structure on the nose.

**Definition 7.** A **pseudo cwf-morphism** from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ is a pair $(F, \sigma)$ where:

— $F : \mathbb{C} \to \mathbb{C}'$ is a functor,
— For each context $\Gamma$ in $\mathbb{C}$, $\sigma_\Gamma$ is a **Fam**-morphism from $T\Gamma$ to $T'F\Gamma$. We will write $\sigma_\Gamma(A) : \mathrm{Type}'(F\Gamma)$, where $A : \mathrm{Type}(\Gamma)$, for the type component and $\sigma_\Gamma^A(a) : F\Gamma \vdash' \sigma_\Gamma(A)$, where $a : \Gamma \vdash A$, for the term component of this morphism.

The following preservation properties must be satisfied:

— Substitution is preserved: For each context $\delta : \Delta \to \Gamma$ in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$, there is an isomorphism of types $\theta_{A,\delta} : \sigma_\Gamma(A)[F\delta] \to \sigma_\Delta(A[\delta])$ such that substitution in terms is also preserved, that is, $\sigma_\Delta^{A[\gamma]}(a[\gamma]) =_{\theta_{A,\gamma}} \sigma_\Gamma^A(a)[F\gamma]$.
— The terminal object is preserved: $F[]$ is terminal, let $\iota : [] \to F[]$ be the isomosrphism.
— Context comprehension is preserved: The context $F(\Gamma \cdot A)$, along with the projections $F(\mathrm{p}_{\Gamma,A})$ and $\{\theta_{A,\mathrm{p}}^{-1}\}(\sigma_{\Gamma \cdot A}^{A[\mathrm{p}]}(\mathrm{q}_{\Gamma,A}))$, is a context comprehension of $F\Gamma$ and $\sigma_\Gamma(A)$. Note that the universal property of context comprehension provides a unique isomorphism $\rho_{\Gamma,A} : F(\Gamma \cdot A) \to F\Gamma \cdot \sigma_\Gamma(A)$ which preserves projections in the following sense:

$$F(\mathrm{p}_A) = \mathrm{p}_{\sigma_\Gamma(A)}\rho_{\Gamma,A} \tag{a}$$

$$\sigma_{\Gamma \cdot A}^{A[\mathrm{p}]}(\mathrm{q}_A) = \{\theta_{A,\mathrm{p}}\}(\mathrm{q}_{\sigma_\Gamma A}[\rho_{\Gamma,A}]) \tag{b}$$

These data must satisfy naturality and coherence laws which amount to the fact that if we extend $\sigma_\Gamma$ to a functor $\boldsymbol{\sigma}_\Gamma : \boldsymbol{T}(\Gamma) \to \boldsymbol{T'}F(\Gamma)$, then $\boldsymbol{\sigma}$ is a pseudonatural transformation from $\boldsymbol{T}$ to $\boldsymbol{T'}F$. This functor is defined by $\boldsymbol{\sigma}_\Gamma(A) = \sigma_\Gamma(A)$ on an object $A$ and $\boldsymbol{\sigma}_\Gamma(f) = \rho_{\Gamma,B}F(f)\rho_{\Gamma,A}^{-1}$ on a morphism $f : A \to B$.

More explicitely, pseudonaturality of $\boldsymbol{\sigma}$ amounts to the following coherence and naturality laws.

— *Identity.* For all $A \in \mathrm{Type}(\Gamma)$, we have $\theta_{A,\mathrm{id}} = \mathrm{id}_{F\Gamma\sigma_\Gamma(A)}$,
— *Coherence.* For all $\delta : \Xi \to \Delta$ and $\gamma : \Delta \to \Gamma$, the following diagram commutes.

$$\begin{tikzcd}
F\Xi\cdot\sigma_\Gamma(A)[F(\gamma\delta)] \arrow[rr, "\theta_{A,\gamma\delta}"] \arrow[dr, "\boldsymbol{T'}(F\delta)(\theta_{A,\gamma})"'] & & F\Xi\cdot\sigma_\Xi(A[\gamma\delta]) \\
& F\Xi\cdot\sigma_\Delta(A[\gamma])[F(\delta)] \arrow[ur, "\theta_{A[\gamma],\delta}"'] &
\end{tikzcd}$$

— *Naturality.* For all $\delta : \Delta \to \Gamma$ in $\mathbb{C}$, $A, B \in \mathrm{Type}(\Gamma)$ and $f : A \to B$ in $\boldsymbol{T}(\Gamma)$, the following diagram commutes in $\boldsymbol{T'}(F\Delta)$.

$$\begin{tikzcd}
\sigma_\Gamma(A)[F\delta] \arrow[r, "\theta_{A,\delta}"] \arrow[d, "\boldsymbol{T'}(F\delta)(\boldsymbol{\sigma}_\Gamma(f))"] & \sigma_\Delta(A[\delta]) \arrow[d, "\boldsymbol{\sigma}_\Delta(\boldsymbol{T}(\delta)(f))"] \\
\sigma_\Gamma(B)[F\delta] \arrow[r, "\theta_{B,\delta}"] & \sigma_\Delta(B[\delta])
\end{tikzcd}$$

From this definition we can prove that all cwf structure is preserved.

**Proposition 4.** All pseudo cwf-morphisms $(F, \sigma)$ from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ preserve substitution extension in the sense that, if $\delta : \Delta \to \Gamma$ in $\mathbb{C}$ and $a : \Delta \vdash A[\delta]$, then

$$F(\langle\delta, a\rangle) = \rho_{\Gamma,A}^{-1}\langle F\delta, \{\theta_{A,\delta}^{-1}\}(\sigma_\Delta^{A[\delta]}(a))\rangle$$

*Proof.* The required equality boils down to the following two equations.

$$\begin{aligned}
\mathrm{p}\rho_{\Gamma,A}F(\langle\delta, a\rangle) &= F\delta \\
\mathrm{q}[\rho_{\Gamma,A}F(\langle\delta, a\rangle)] &= \{\theta_{A,\delta}^{-1}\}(\sigma_\Delta^{A[\delta]}(a))
\end{aligned}$$

The proof of the first equality is straightforward, using (a):

$$\begin{aligned}
\mathrm{p}\rho_{\Gamma,A}F(\langle\delta, a\rangle) &= F(\mathrm{p})F(\langle\delta, a\rangle) \\
&= F\delta
\end{aligned}$$

However, the proof of the second is far more subtle and relies on many properties of pseudo cwf-morphisms and cwf combinators:

$$\begin{array}{rcl}
\mathrm{q}[\rho_{\Gamma,A}F(\langle\delta,a\rangle)] & =_1 & \{\theta^{-1}_{A,\mathrm{p}}\}(\sigma^{A[\mathrm{p}]}_{\Gamma A}(\mathrm{q}))[F(\langle\delta,a\rangle)] \\
& =_2 & \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle \mathrm{id}, \sigma^{A[\mathrm{p}]}_{\Gamma A}(\mathrm{q})\rangle][F(\langle\delta,a\rangle)] \\
& = & \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle\delta,a\rangle), \sigma^{A[\mathrm{p}]}_{\Gamma A}(\mathrm{q})[F(\langle\delta,a\rangle)]\rangle] \\
& =_3 & \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle\delta,a\rangle), \{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(\mathrm{q}[\langle\delta,a\rangle]))\rangle] \\
& = & \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle\delta,a\rangle), \{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\
& = & \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle\delta,a\rangle)\mathrm{p},\mathrm{q}\rangle]\rangle\langle \mathrm{id}, \{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\
& =_4 & \mathrm{q}[\boldsymbol{T'}(F(\langle\delta,a\rangle))(\theta^{-1}_{A,\mathrm{p}})\langle \mathrm{id}, \{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\
& =_2 & \{\boldsymbol{T'}(F(\langle\delta,a\rangle))(\theta^{-1}_{A,\mathrm{p}})\}(\{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))) \\
& =_5 & \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_{\Delta}(a))
\end{array}$$

Equality (1) is by (b), equalities (2) by definition of coercions, equality (3) by preservation of substitution in terms, equality (4) by definition of $\boldsymbol{T'}$, equality (5) by the coherence requirement on $\theta$ and the fact that coercions compose. All the other steps are by simple manipulations on cwf combinators. □

As an aside, recall that for each substitution $\delta : \Delta \to \Gamma$ and type $A \in \mathrm{Type}(\Gamma)$ in a cwf we have the following pullback square:

$$\begin{tikzcd}
\Delta\cdot A[\delta] \arrow[r, "\langle\delta\mathrm{p}_A,\mathrm{q}_A\rangle"] \arrow[d, "\mathrm{p}_{A[\delta]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma\cdot A \arrow[d, "\mathrm{p}_A"] \\
\Delta \arrow[r, "\delta"] & \Gamma
\end{tikzcd}$$

It is no surprise that these pullback squares will play an important role in the technical development, and it will be particularly useful to know how they are preserved by a pseudo cwf-morphism $(F,\sigma)$ from $(\mathbb{C},T)$ to $(\mathbb{C}',T')$. From equality (a) we know that the first projection is preserved, so the preservation of the pullback square above amounts to the equality $F(\langle\delta\mathrm{p},\mathrm{q}\rangle) = \rho^{-1}_{\Gamma,A}\langle F(\delta)\mathrm{p},\mathrm{q}\rangle\theta^{-1}_{A,\delta}\rho_{\Delta,A[\delta]}$, which is established by a direct calculation (see Lemma 8 in the Appendix).

In a cwf, terms can be converted into sections of display maps and vice versa. From the definition of pseudo cwf-morphisms, it follows that they behave coherently with respect to this conversion.

**Proposition 5.** If $(F,\sigma)$ is a pseudo cwf-morphism from $(\mathbb{C},T)$ to $(\mathbb{C}',T')$, then its action on terms is determined by its action on sections: for all $a \in \Gamma \vdash A$

$$\sigma^A_\Gamma(a) = \mathrm{q}[\rho_{\Gamma,A}F(\langle \mathrm{id},a\rangle)]$$

*Proof.* This follows from preservation of substitution extension (Proposition 4):

$$F(\langle \mathrm{id}, a\rangle) = \rho^{-1}_{\Gamma,A}\langle \mathrm{id}, \{\theta^{-1}_{A,\mathrm{id}}\}\sigma^A_\Gamma(a)\rangle$$

but $\theta_{A,\mathrm{id}} = \mathrm{id}$ by coherence of $\theta$, hence the result is proved. □

**Proposition 6.** Pseudo cwf-morphisms are stable under composition.

*Proof.* If $(F,\sigma) : (\mathbb{C}_0, T_0) \to (\mathbb{C}_1, T_1)$ and $(G,\tau) : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ are two pseudo cwf-morphisms, we define their composition as $(GF, \tau\sigma)$ where:

$$\begin{array}{rcl} (\tau\sigma)_\Gamma(A) & = & \tau_{F\Gamma}(\sigma_\Gamma(A)) \\ (\tau\sigma)^A_\Gamma(a) & = & \tau^{\sigma_\Gamma(A)}_{F\Gamma}(\sigma^A_\Gamma(a)) \end{array}$$

If the other components of $(F,\sigma)$ are denoted by $\theta^F, \rho^F$ and those of $(G,\tau)$ by $\theta^G, \rho^G$, we define:

$$\theta_{A,\delta} = \boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\theta^G_{\sigma_\Gamma(A),F\delta}$$

All the components are now defined, and we can show that the conditions hold.

— *Preservation of substitution on terms.* Direct calculation, if $a : \Gamma \vdash A$ and $\delta : \Delta \to \Gamma$ in $\mathbb{C}_0$.

$$\begin{array}{rcl} (\tau\sigma)^{A[\delta]}_\Delta(a[\delta]) & =_1 & \tau^{\sigma_\Delta(A[\delta])}_{F\Delta}(\sigma^{A[\delta]}_\Delta(a[\delta])) \\ & =_2 & \tau^{\sigma_\Delta(A[\delta])}_{F\Delta}(\{\theta^F_{A,\delta}\}(\sigma^A_\Gamma(a)[F\delta])) \\ & =_3 & \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\langle \mathrm{id}, \{\theta^F_{A,\delta}\}(\sigma^A_\Gamma(a)[F\delta])\rangle)] \\ & =_4 & \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\langle \mathrm{id}, \mathrm{q}[\theta^F_{A,\delta}\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle]\rangle)] \\ & =_5 & \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\theta^F_{A,\delta}\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)] \\ & =_6 & \mathrm{q}[\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)] \\ & = & \mathrm{q}[\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\langle \mathrm{p}, \mathrm{q}\rangle\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)] \\ & =_7 & \mathrm{q}[\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\langle \mathrm{id}, \mathrm{q}[\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)]\rangle] \\ & =_4 & \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\mathrm{q}[\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)]) \\ & =_3 & \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\tau^{\sigma_\Gamma(A)[F\delta]}_{F\Delta}(\sigma^A_\Gamma(a)[F\delta])) \\ & =_2 & \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\{\theta^G_{\sigma_\Gamma(A),F\delta}\}(\tau^{\sigma_\Gamma(A)}_{F\Gamma}(\sigma^A_\Gamma(a))[GF\delta])) \\ & =_8 & \{\theta_{A,\delta}\}(\tau^{\sigma_\Gamma(A)}_{F\Gamma}(\sigma^A_\Gamma(a))[GF\delta]) \\ & =_1 & \{\theta_{A,\delta}\}((\tau\sigma)^A_\Gamma(a)[GF\delta]) \end{array}$$

Equalities annotated by (1) come from the definition of $\tau\sigma$, (2) is preservation of substitution for $\sigma$ or $\tau$, (3) is Proposition 5, (4) is by definition of coercions, (5) uses $\mathrm{p}\theta^F_{A,\delta} = \mathrm{p}$ and basic manipulations with cwf combinators, (6) is by definition of $\boldsymbol{\tau}$, (7) uses preservation of p by $(G,\tau)$ and basic manipulations with cwf combinators, and (8) is by definition of $\theta$.

— *Preservation of the terminal object.* Trivial from the preservation of the terminal object by $F$ and $G$.

— *Preservation of context comprehension.* Using preservation of context comprehension from $(F, \sigma)$ and $(G, \tau)$ we define:

$$GF(\Gamma \cdot A) \xrightarrow{G(\rho^F_{\Gamma,A})} G(F\Gamma \cdot \sigma_\Gamma(A)) \xrightarrow{\rho^G_{F\Gamma,\sigma_\Gamma A}} GF\Gamma \cdot (\tau\sigma)_\Gamma(A)$$

Isomorphisms compose, and hence $GF(\Gamma \cdot A)$ is also a context comprehension of $GF\Gamma$ and $(\tau\sigma)_\Gamma(A)$. We must check that the corresponding projections are those required by the definition. This is easy for the first projection:

$$\begin{array}{rcl} \mathrm{p}\rho^G_{F\Gamma,\sigma_\Gamma A} G(\rho^F_{\Gamma,A}) & = & G(\mathrm{p})G(\rho^F_{\Gamma,A}) \\ & = & GF\mathrm{p} \end{array}$$

But more intricate for the second.

$$\begin{array}{rcl} \mathrm{q}[\rho^G_{F\Gamma,\sigma_\Gamma A} G(\rho^F_{\Gamma,A})] & =_1 & \{(\theta^G_{\sigma_\Gamma A,\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[\mathrm{p}]}_{F\Gamma \sigma_\Gamma A}(\mathrm{q}))[G(\rho^F_{\Gamma,A})] \\ & =_2 & \{(\theta^G_{\sigma_\Gamma A,\mathrm{p}})^{-1}\}(\{(\theta^G_{\sigma_\Gamma(A)[\mathrm{p}],\rho^F_{\Gamma,A}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma \cdot A)}(\mathrm{q}[\rho^F_{\Gamma,A}]))) \\ & =_3 & \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma \cdot A)}(\mathrm{q}[\rho^F_{\Gamma,A}])) \\ & =_4 & \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma \cdot A)}(\{(\theta^F_{A,\mathrm{p}})^{-1}\}(\sigma^{A[\mathrm{p}]}_{\Gamma \cdot A}(\mathrm{q})))) \\ & =_5 & \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\{\boldsymbol{\tau}_{F(\Gamma \cdot A)}((\theta^F_{A,\mathrm{p}})^{-1})\}(\tau^{\sigma_{\Gamma \cdot A}(A[\mathrm{p}])}_{F(\Gamma \cdot A)}(\sigma^{A[\mathrm{p}]}_{\Gamma \cdot A}(\mathrm{q})))) \\ & =_6 & \{\theta^{-1}_{A,[p]}\}((\tau\sigma)^{A[\mathrm{p}]}_{\Gamma \cdot A}(q)) \end{array}$$

Here (1) is Equation (b) on $\rho^G$, (2) is preservation of substitution on terms, (3) is coherence for $\theta^G$, (4) is Equation (b) of the second projection by $\rho^F$, (5) is Lemma 9 (see Appendix), and (6) is by definition of $\theta$ and $\tau\sigma$.

Finally, by unfolding the definitions we can conclude that for all contexts $\Gamma$

$$(\boldsymbol{\tau\sigma})_\Gamma = \boldsymbol{\tau}_{F\Gamma} \circ \boldsymbol{\sigma}_\Gamma$$

Hence the necessary coherence and naturality conditions amount to the stability of pseudonatural transformations under composition. □

### 3.2. *Preservation of structure*

If cwfs support other structure, we need to define what it means that cwf-morphisms preserve this extra structure up to isomorphism.

3.2.1. *Preservation of $\Sigma$-types.* First, we consider $\Sigma$-types. Since the isomorphism $(\Gamma \cdot A) \cdot B \cong \Gamma \cdot \Sigma(A, B)$ holds in an arbitrary cwf which supports $\Sigma$-types, it follows that pseudo cwf-morphisms preserve $\Sigma$-types, since they preserve context comprehension. More precisely, we have

**Proposition 7.** A pseudo cwf-morphism $(F, \sigma)$ from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$, where both cwfs support $\Sigma$-types, also preserves them in the sense that there is an isomorphism:

$$\sigma_\Gamma(\Sigma(A, B)) \cong_{s_{A,B}} \Sigma(\sigma_\Gamma(A), \sigma_{\Gamma \cdot A}(B)[\rho^{-1}_{\Gamma,A}])$$

such that projections are preserved up to isomorphism. For any term $\Gamma \vdash c : \Sigma(A, B)$, or terms $a : \Gamma \vdash A$ and $b : \Gamma \vdash B[\langle \mathrm{id}, a\rangle]$.

$$\begin{array}{rcl} \sigma_\Gamma^A(\pi_1(c)) & = & \pi_1(\{s_{A,B}\}(\sigma_\Gamma^{\Sigma(A,B)}(c))) \\ \sigma_\Gamma^{B[\langle \mathrm{id}, \pi_1(c)\rangle]}(\pi_2(c)) & = & \{\theta_{B,\langle \mathrm{id}, \pi_1(c)\rangle}\}(\pi_2(\{s_{A,B}\}(\sigma_\Gamma^{\Sigma(A,B)}(c)))) \\ \sigma_\Gamma^{\Sigma(A,B)}(\mathrm{pair}(a, b)) & = & \{s_{A,B}^{-1}\}(\mathrm{pair}(\sigma_\Gamma^A(a), \{\theta_{B,\langle \mathrm{id}, a\rangle}^{-1}\}(\sigma_\Gamma^{B[\langle id, a\rangle]}(b)))) \end{array}$$

*Proof.* First we have the two following substitutions:

$$\begin{array}{rcl} \langle \mathrm{pp}, \mathrm{pair}(\mathrm{q}[\mathrm{p}], \mathrm{q})\rangle & : & \Gamma\cdot A\cdot B \to \Gamma\cdot\Sigma(A, B) \\ \langle \mathrm{p}, \pi_1(\mathrm{q}), \pi_2(\mathrm{q})\rangle & : & \Gamma\cdot\Sigma(A, B) \to \Gamma\cdot A\cdot B \end{array}$$

It is straightforward to check that they define an isomorphism $\gamma_{A,B} : \Gamma\cdot A\cdot B \to \Gamma\cdot\Sigma(A, B)$. Note that from $\gamma_{A,B}$, the projections can be recovered by, for $c : \Gamma \vdash \Sigma(A, B)$:

$$\pi_1(c) = \mathrm{q}[\mathrm{p}\gamma_{A,B}^{-1}\langle \mathrm{id}, c\rangle] \qquad \pi_2(c) = \mathrm{q}[\gamma_{A,B}^{-1}\langle \mathrm{id}, c\rangle]$$

Let us write $\gamma'_{A,B}$ for the similar isomorphism on objects of the category $\mathbb{C}'$. We now form the isomorphism $s_{A,B}$ as the following composition of isomorphisms:

$$\begin{array}{rcl} F\Gamma\cdot\sigma_\Gamma(\Sigma(A, B)) & \cong_{\rho^{-1}_{\Gamma,\Sigma(A,B)}} & F(\Gamma\cdot\Sigma(A, B)) \\ & \cong_{F(\gamma^{-1}_{A,B})} & F(\Gamma\cdot A\cdot B) \\ & \cong_{\rho_{\Gamma\cdot A,B}} & F(\Gamma\cdot A)\cdot\sigma_{\Gamma\cdot A}(B) \\ & \cong_{\langle \rho_{\Gamma,A}\mathrm{p},\mathrm{q}\rangle} & F\Gamma\cdot\sigma_\Gamma(A)\cdot\sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}] \\ & \cong_{\gamma'_{\sigma_\Gamma(A),\sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}]}} & F\Gamma\cdot\Sigma(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}]) \end{array}$$

Preservation of projections and pairing follows from an intricate calculation on cwf combinators. We omit it to avoid making the development heavier, since we will not use it. □

3.2.2. *Preservation of identity types and democracy.*

**Definition 8.** Let $(F, \sigma)$ be a pseudo cwf-morphism between cwfs $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ which support identity types and democracy, respectively.

- — $(F, \sigma)$ *preserves identity types* provided $\sigma_\Gamma(\mathrm{I}_A(a, a')) \cong \mathrm{I}_{\sigma_\Gamma(A)}(\sigma_\Gamma^A(a), \sigma_\Gamma^A(a'))$;
- — $(F, \sigma)$ *preserves democracy* provided $\sigma_{[]}(\overline{\Gamma}) \cong_{d_\Gamma} \overline{F\Gamma}[\langle\rangle]$, and the following diagram commutes:

$$\begin{tikzcd} F\Gamma \arrow[rr, "F\gamma_\Gamma"] \arrow[d, "\gamma_{F\Gamma}"'] & & F([]\cdot\overline{\Gamma}) \arrow[d, "\rho_{[],\overline{\Gamma}}"] \\ {[]\cdot\overline{F\Gamma}} & F[]\cdot\overline{F\Gamma}[\langle\rangle] \arrow[l, "{\langle\langle\rangle,\mathrm{q}\rangle}"'] & F[]\cdot\sigma_{[]}(\overline{\Gamma}) \arrow[l, "d_\Gamma"'] \end{tikzcd}$$

**Proposition 8.** If $(F, \sigma) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ and $(G, \tau) : (\mathbb{C}', T') \to (\mathbb{C}'', T'')$ preserve identity types (resp. democracy), then so does $(G, \tau) \circ (F, \sigma)$.

*Proof.* This is trivial for identity types. For democracy, we must check that the isomorphism $d^{GF}_{\Gamma} : GF[]\cdot(\tau\sigma)_{[]}(\overline{\Gamma}) \to GF[]\cdot\overline{GF\Gamma}[\langle\rangle]$ defined by

$$\begin{aligned} d^{GF}_{\Gamma} = \ & GF[]\cdot\tau_{F[]}(\sigma_{[]}(\overline{\Gamma})) \xrightarrow{(\rho^G_{F[],\sigma_{[]}\overline{\Gamma}})^{-1}} G(F[]\cdot\sigma_{[]}\overline{\Gamma}) \xrightarrow{G(d^F_{\Gamma})} G(F[]\cdot\overline{F\Gamma}[\langle\rangle]) \\ & \xrightarrow{G(\langle\langle\rangle,\mathrm{q}\rangle)} G([]\cdot\overline{F\Gamma}) \xrightarrow{\rho^G_{[],\overline{F\Gamma}}} G[]\cdot\tau_{[]}\overline{F\Gamma} \\ & \xrightarrow{d^G_{F\Gamma}} G[]\cdot\overline{GF\Gamma}[\langle\rangle] \end{aligned}$$

satisfies the coherence law. This follows by simple diagram chase. □

3.2.3. *Preservation of* Π*-types.*

**Definition 9.** Let $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ be cwfs supporting Π-types, and $(F, \sigma)$ a pseudo cwf-morphism. Then $(F, \sigma)$ preserves Π-types iff for each types $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma\cdot A)$ there is an isomorphism in $\boldsymbol{T'}(\Gamma)$:

$$\sigma_\Gamma(\Pi(A, B)) \to_{i_{A,B}} \Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])$$

such that for any substitution $\delta : \Delta \to \Gamma$, for any terms $c : \Delta \vdash \Pi(A, B)[\delta]$ and $a : \Gamma \vdash A[\delta]$, we have:

$$\sigma^{B[\langle\delta,a\rangle]}_{\Gamma}(\mathrm{ap}(c, a)) = \{\theta_{B,\langle\delta,a\rangle}\}(\mathrm{ap}(\{\boldsymbol{T'}(F\delta)(i_{A,B})\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma^{\Pi(A,B)}_{\Gamma}(c), \{\theta^{-1}_{A,\delta}\}(\sigma^{A}_{\Gamma}(a)))))$$

Note that from this definition, it follows that abstraction is preserved as well, in the following sense:

$$\sigma^{\Pi(A,B)}_{\Gamma}(\lambda(b)) = \{i^{-1}_{A,B}\}(\lambda(\sigma^{B}_{\Gamma\cdot A}(b)[\rho^{-1}_{\Gamma,A}]))$$

This property will not be used directly and follows from later developments (Lemma 2), so we omit the direct proof.

Likewise, later developments (Lemma 2) will entail that preservation of Π-types is stable under composition of pseudo cwf-morphisms.

## 3.3. *Pseudo Cwf-Transformations*

**Definition 10 (Pseudo cwf-transformation).** Let $(F, \sigma)$ and $(G, \tau)$ be two cwf-morphisms from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$. A *pseudo cwf-transformation* from $(F, \sigma)$ to $(G, \tau)$ is a pair $(\phi, \psi)$ where $\phi : F \xrightarrow{\bullet} G$ is a natural transformation, and for each $\Gamma$ in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$, a morphism $\psi_{\Gamma,A} : \sigma_\Gamma(A) \to \tau_\Gamma(A)[\phi_\Gamma]$ in $\boldsymbol{T'}(F\Gamma)$, natural in $A$ and such that the following diagram commutes:

$$\begin{tikzcd} \sigma_\Gamma(A)[F\delta] \arrow[r, "\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A})"] \arrow[d, "\theta_{A,\delta}"] & \tau_\Gamma(A)[\phi_\Gamma F(\delta)] \arrow[d, "\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})"] \\ \sigma_\Delta(A[\delta]) \arrow[r, "\psi_{\Delta,A[\delta]}"'] & \tau_\Delta(A[\delta])[\phi_\Delta] \end{tikzcd}$$

where $\theta$ and $\theta'$ are the isomorphisms witnessing preservation of substitution in types in the definition of pseudo cwf-morphism.

Pseudo cwf-transformations can be composed both vertically (denoted by $(\phi', \psi') \bullet (\phi, \psi)$) and horizontally (denoted by $(\phi', \psi')(\phi, \psi)$), and these compositions are associative and satisfy the interchange law. Note that just as coherence and naturality laws for pseudo cwf-morphisms ensure that they give rise to pseudonatural transformations (hence morphisms of indexed categories) $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$, this definition means that pseudo cwf-transformations from $(F, \sigma)$ to $(F, \tau)$ correspond to modifications from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$.

We note in passing that pseudo cwf-morphisms preserve coercions, in the sense that if $(F, \sigma)$ is a pseudo cwf-morphism from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ and for all morphism $f : A \to B$ in $\boldsymbol{T}(\Gamma)$, for each term $a : \Gamma \vdash A$ we have that $\sigma_\Gamma^B(\{f\}(a)) = \{\boldsymbol{\sigma}_\Gamma(f)\}(\sigma_\Gamma^A(a))$ (see Lemma 9 in the Appendix).

### 3.4. *2-Categories of Cwfs with Extra Structure*

As a consequence of the preservation properties in Proposition 8 we have several different 2-categories of structure-preserving pseudo cwf-morphisms.

**Definition 11.** Let $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma}$ be the 2-category of small democratic categories with families which support extensional identity types and $\Sigma$-types. The 1-cells are cwf-morphisms preserving democracy and extensional identity types (and $\Sigma$-types automatically) and the 2-cells are pseudo cwf-transformations.

Moreover, let $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma\Pi}$ be the sub-2-category of $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma}$ where also $\Pi$-types are supported and preserved.

## 4. Forgetting Types and Terms

In this section we provide the first components of our biequivalences, which will be *forgetful* 2-functors:

**Proposition 9.** The forgetful 2-functors

$$\begin{array}{rcl} U & : & \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma} \to \mathbf{FL} \\ U & : & \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma\Pi} \to \mathbf{LCC} \end{array}$$

defined as follows on 0-, 1-, and 2-cells

$$\begin{array}{rcl} U(\mathbb{C}, T) & = & \mathbb{C} \\ U(F, \sigma) & = & F \\ U(\phi, \psi) & = & \phi \end{array}$$

are well-defined.

Here, **FL** and **LCC** are respectively the 2-categories of categories with finite limits and locally cartesian closed categories defined below. By definition, $U$ already is a 2-functor from **CwF** to **Cat**.

We already proved as corollaries of Proposition 3 that if $(\mathbb{C}, T)$ supports $\Sigma$-types, identity types and democracy, then $\mathbb{C}$ has finite limits; and if $(\mathbb{C}, T)$ also supports $\Pi$-

types, then $\mathbb{C}$ is an lccc. Hence $U$ sends a 0-cell in $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ to a 0-cell in in $\mathbf{FL}$ and a 0-cell in $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ to a 0-cell in $\mathbf{LCC}$.

For 1-cells we shall prove in Proposition 10 that if $(F, \sigma)$ preserves identity types and democracy, then $F$ preserves finite limits; and in Proposition 12 that if $(F, \sigma)$ also preserves $\Pi$-types then $F$ preserves the locally cartesian closed structure.

There is nothing to prove for 2-cells.

### 4.1. *Preservation of finite limits*

We shall prove that $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ is biequivalent to the following 2-category.

**Definition 12.** Let $\mathbf{FL}$ be the 2-category of small categories with finite limits (left exact categories). The 1-cells are functors preserving finite limits (up to isomorphism) and the 2-cells are natural transformations.

$\mathbf{FL}$ is a sub(2-)category of the 2-category of categories: we do not provide a choice of finite limits.

Recall that in the presence of a terminal object, finite limits can be obtained by pullbacks or by binary products and equalizers. That a functor $F : \mathbb{C} \to \mathbb{D}$ preserves finite limits means it preserves the limiting cone, *i.e.* the image of any universal cone is a universal cone. Alternatively $F$ preserves the terminal object and the image of any pullback diagram is a pullback diagram (or the image of a product (resp. equalizer) diagram is a product (resp. equalizer) diagram). By the universal property of limits, it suffices to check that any well-chosen universal cone is universal.

Let $(F, \sigma)$ be a pseudo cwf-morphism between $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ which are democratic cwfs supporting $\Sigma$-types and identity types. We wish to prove that $F$ preserves pullbacks. We remark first that we already know that $F$ preserves *some* pullbacks. Indeed by the cwf structure, for each context $\Gamma$, type $A \in \mathrm{Type}(\Gamma)$, and substitution $\gamma : \Delta \to \Gamma$, we have the *chosen* pullback:

$$\begin{tikzcd}
\Delta \cdot A[\gamma] \arrow[r, "\langle \gamma \mathrm{p}, \mathrm{q} \rangle"] \arrow[d, "\mathrm{p}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \cdot A \arrow[d, "\mathrm{p}"] \\
\Delta \arrow[r, "\gamma"] & \Gamma
\end{tikzcd}$$

The fact that this chosen pullback is preserved follows easily from the fact that $(F, \sigma)$ preserves substitution and context comprehension. However, we need to show that $F$ preserves arbitrary pullbacks, not only those of display maps along substitutions. In Section 2.7, we already showed that any substitution can be (up to isomorphism) presented as a display map using the inverse image construction, allowing in particular to construct arbitrary pullbacks. For these pullbacks to be preserved, the key observation is the following lemma, showing that the inverse image construction is preserved.

**Lemma 1 (Preservation of inverse image).** Let $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ be cwfs supporting democracy, $\Sigma$-types and identity types and let $(F, \sigma)$ be a pseudo cwf-morphism preserving them. Moreover, suppose that $\delta : \Delta \to \Gamma$ is a morphism in $\mathbb{C}$, then there is an isomorphism in $\mathbb{C}'$:

$$\zeta : F(\Gamma\cdot\mathrm{Inv}(\delta)) \quad\cong\quad F\Gamma\cdot\mathrm{Inv}(F\delta)$$

such that $\mathrm{p}\zeta = F(\mathrm{p})$.

*Proof.* We first remark that the type constructors $\Sigma$ and I preserves isomorphisms, in a sense made formal in Lemma 10 in the Appendix. From that, the isomorphism boils down to an intricate calculation on cwf combinators, detailed in the appendix. $\square$

Now, we can show that pseudo cwf-morphisms preserving democracy, $\Sigma$-types and identity types preserve finite limits. In fact we prove more: the property turns out to be an equivalence.

**Proposition 10.** Let $(F, \sigma)$ be a pseudo cwf-morphism preserving democracy between $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ supporting democracy, $\Sigma$-types and identity types. Then $(F, \sigma)$ preserves identity types if and only if $F$ preserves finite limits.

*Proof. If.* Let $a, a' : \Gamma \vdash A$ be two terms in $(\mathbb{C}, T)$. Then, $\mathrm{I}_A(a, a') \in \mathrm{Type}(\Gamma)$ is such that $\mathrm{p}_{\mathrm{I}_A(a,a')}$ is an equalizer of $\langle \mathrm{id}, a\rangle, \langle \mathrm{id}, a'\rangle : \Gamma \to \Gamma\cdot A$. If $F$ preserves equalizers then $F(\mathrm{p}_{\mathrm{I}_A(a,a')}) : F(\Gamma\cdot\mathrm{I}_A(a, a')) \to F(\Gamma)$ is an equalizer of $F(\langle \mathrm{id}, a\rangle)$ and $F(\langle \mathrm{id}, a'\rangle)$. By preservation of cwf structure, it immediately follows that $F(\mathrm{p}_{\mathrm{I}_A(a,a')})$ must also be an equalizer of $\langle \mathrm{id}, \sigma^A_\Gamma(a)\rangle$ and $\langle \mathrm{id}, \sigma^A_\Gamma(a')\rangle$. But since $(\mathbb{C}', T')$ supports identity types, these already have an equalizer

$$\mathrm{p} : F\Gamma\cdot\mathrm{I}_{\sigma_\Gamma(A)}(\sigma^A_\Gamma(a), \sigma^A_\Gamma(a'))$$

It follows that $\sigma_\Gamma(\mathrm{I}_A(a, a')) \cong \mathrm{I}_{\sigma_\Gamma(A)}(\sigma^A_\Gamma(a), \sigma^A_\Gamma(a'))$.

*Only if.* Suppose $(F, \sigma)$ preserves identity types. We already know that $F$ preserves the terminal object. To prove that it preserves pullbacks, take $f : \Delta \to \Gamma$ and $f' : \Delta' \to \Gamma$. As explained in Section 2.7, we know that $f$ (resp. $f'$) is isomorphic to $\mathrm{p}_{\mathrm{Inv}(f)}$ (resp. $\mathrm{p}_{\mathrm{Inv}(f')}$) in $\mathbb{C}/\Gamma$. Therefore to check that $F$ preserves pullbacks, it suffices to check the chosen pullback:

$$\begin{tikzcd}
\Gamma\cdot\mathrm{Inv}(f)\cdot\mathrm{Inv}(f')[\mathrm{p}] \arrow[r, "\langle \mathrm{pp},\mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma\cdot\mathrm{Inv}(f') \arrow[d, "\mathrm{p}"] \\
\Gamma\cdot\mathrm{Inv}(f) \arrow[r, "p"] & \Gamma
\end{tikzcd}$$

is preserved. But by Lemma 1 and preservation of substitution and context comprehension, the image of this pullback is isomorphic to the following pullback:

$$\begin{tikzcd}
F\Gamma\cdot\mathrm{Inv}(Ff)\cdot\mathrm{Inv}(Ff')[\mathrm{p}] \arrow[r, "\langle \mathrm{pp},\mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"] \arrow[dr, phantom, "\lrcorner", very near start] & F\Gamma\cdot\mathrm{Inv}(Ff') \arrow[d, "\mathrm{p}"] \\
F\Gamma\cdot\mathrm{Inv}(Ff) \arrow[r, "p"] & F\Gamma
\end{tikzcd}$$

which concludes the proof. $\square$

### 4.2. *Preservation of locally cartesian closed structure*

We shall prove that $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$ is biequivalent to the following 2-category.

**Definition 13.** Let **LCC** be the 2-category of small locally cartesian closed categories. The 1-cells are functors preserving local cartesian closed structure (up to isomorphism), and the 2-cells are natural transformations.

Similarly to **FL** we do not assume chosen structure in **LCC**: it is a sub(2-)category of **FL**.

For completeness, let us recall what is meant by preservation of the locally cartesian closed structure.

A locally cartesian closed category can be defined as a category with a terminal object such that every slice category is cartesian closed. Equivalently, it is a category with finite limits such that for any $f : A \to B$, the pullback functor $f^* : \mathbb{C}/B \to \mathbb{C}/A$ has a right adjoint $\Pi_f : \mathbb{C}/A \to \mathbb{C}/B$. As usual, the existence of this right adjoint can be expressed by a universal property. If $g : A \to B$ and $f : B \to C$ are morphisms, define a dependent product of $g$ along $f$ is a diagram of the form:

$$\begin{tikzcd}
 & P \arrow[r] \arrow[d] \arrow[dl, bend right, "ev"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, "\Pi_f(g)"] \\
A \arrow[r, "g"] & B \arrow[r, "f"] & C
\end{tikzcd}$$

which is universal among any such diagram over $g$ and $f$, as described below.

$$\begin{tikzcd}
 & P' \arrow[r] \arrow[d, dotted] \arrow[ddl, bend right] \arrow[dd, bend right] \arrow[dr, phantom, "\lrcorner", very near start] & D' \arrow[d, dotted] \arrow[dd, bend right] \\
 & P \arrow[r] \arrow[d] \arrow[dl, bend right] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d] \\
A \arrow[r] & B \arrow[r] & C
\end{tikzcd}$$

Therefore, a category with finite limits $\mathbb{C}$ is locally cartesian closed if for any $g : A \to B$ and $f : B \to C$, there is a dependent product diagram. A functor $F : \mathbb{C} \to \mathbb{D}$ preserves dependent product if the image of a dependent product diagram is a dependent product diagram. Finally, a functor $F : \mathbb{C} \to \mathbb{D}$ preserves the lccc structure iff it preserves finite limits and dependent products — this is equivalent to the fact that $F$ preserves the terminal object and for all object $A$ in $\mathbb{C}$, the functor $F/A : \mathbb{C}/A \to \mathbb{C}/FA$ preserves cartesian closure.

We wish now to prove that $(F, \sigma)$ preserves $\Pi$-types if and only if $F$ preserves dependent products. The first step is to notice that $\Pi$-types in a cwf $(\mathbb{C}, T)$ naturally equip the base category $\mathbb{C}$ with dependent products diagrams along display maps.

**Proposition 11.** Let $(\mathbb{C}, T)$ be a cwf supporting $\Pi$-types, let $\Gamma$ be a context in $\mathbb{C}$, let $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, then the following diagram is an dependent product diagram, where $ev_{A,B} = \langle \mathrm{p}_{\Pi(A,B)[\mathrm{p}_A]}, \mathrm{ap}(\mathrm{q}_{\Pi(A,B)[\mathrm{p}_A]}, \mathrm{q}_A[\mathrm{p}_{\Pi(A,B)[\mathrm{p}_A]}])\rangle$.

$$\begin{tikzcd}
& \Gamma\cdot A\cdot\Pi(A,B)[\mathrm{p}_A] \arrow[dl, bend right, "ev_{A,B}"'] \arrow[d, "\mathrm{p}_{\Pi(A,B)[\mathrm{p}_A]}"] \arrow[r, "\langle \mathrm{p}_A\mathrm{p}_{\Pi(A,B)[\mathrm{p}_A]}, \mathrm{q}_{\Pi(A,B)[\mathrm{p}_A]}\rangle"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma\cdot\Pi(A,B) \arrow[d, "\mathrm{p}_{\Pi(A,B)}"] \\
\Gamma\cdot A\cdot B \arrow[r, "\mathrm{p}_B"] & \Gamma\cdot A \arrow[r, "\mathrm{p}_A"] & \Gamma
\end{tikzcd}$$

It is referred to as the *chosen* dependent product of $\mathrm{p}_B$ along $\mathrm{p}_A$.

*Proof.* The verification is mostly straightforward: take another (chosen) pullback

$$\begin{tikzcd}
& \Delta\cdot A[\delta] \arrow[dl, bend right, "f"'] \arrow[d, "\langle\delta\mathrm{p},\mathrm{q}\rangle"] \arrow[r, "\mathrm{p}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta \arrow[d, "\delta"] \\
\Gamma\cdot A\cdot B \arrow[r, "\mathrm{p}"] & \Gamma\cdot A \arrow[r, "\mathrm{p}"] & \Gamma
\end{tikzcd}$$

Then necessarily, $f = \langle \delta\mathrm{p}, \mathrm{q}, b\rangle$ for some term $b : \Delta\cdot A[\delta] \vdash B[\langle\delta\mathrm{p},\mathrm{q}\rangle]$. Then, we form $\langle\delta, \lambda(b)\rangle : \Delta \to \Gamma{\cdot}\Pi(A,B)$, from which we obtain $\langle\delta\mathrm{p}, \mathrm{q}, \lambda(b)[\mathrm{p}]\rangle : \Delta{\cdot}A[\delta] \to \Gamma{\cdot}A{\cdot}\Pi(A,B)[\mathrm{p}]$. We check that it satisfies the required equations. First, it is straightforward to establish that $\langle\delta, \lambda(b)\rangle\mathrm{p} = \langle\mathrm{pp},\mathrm{q}\rangle\langle\delta\mathrm{p},\mathrm{q},\lambda(b)[\mathrm{p}]\rangle$ using basic manipulation of cwf combinators. For the other equation, we calculate:

$$\begin{aligned}
ev_{A,B}\langle\delta\mathrm{p},\mathrm{q},\lambda(b)[\mathrm{p}]\rangle &=_1 \langle\mathrm{p}, \mathrm{ap}(\mathrm{q},\mathrm{q}[\mathrm{p}])\rangle\langle\delta\mathrm{p},\mathrm{q},\lambda(b)[\mathrm{p}]\rangle \\
&= \langle\delta\mathrm{p},\mathrm{q},\mathrm{ap}(\lambda(b)[\mathrm{p}],\mathrm{q})\rangle \\
&=_2 \langle\delta\mathrm{p},\mathrm{q},\mathrm{ap}(\lambda(b[\langle\mathrm{pp},\mathrm{q}\rangle]),\mathrm{q})\rangle \\
&=_3 \langle\delta\mathrm{p},\mathrm{q},b[\langle\mathrm{pp},\mathrm{q}\rangle][\langle\mathrm{id},\mathrm{q}\rangle]\rangle \\
&= \langle\delta\mathrm{p},\mathrm{q},b\rangle
\end{aligned}$$

where (1) is by definition of $ev$, (2) is by stability of $\lambda$ under substitution, (3) is by the computation rule of $\Pi$-types, and the rest is by simple manipulations of cwf combinators.

For uniqueness, take another $h : \Delta \to \Gamma\cdot\Pi(A,B)$. Necessarily, $h = \langle\delta, c\rangle$ for some term $c : \Delta \vdash \Pi(A,B)[\delta]$. By hypothesis, by forming $\langle\delta\mathrm{p},\mathrm{q},c[\mathrm{p}]\rangle : \Delta\cdot A[\delta] \to \Gamma\cdot A\cdot\Pi(A,B)[\mathrm{p}]$ we get a substitution such that $ev\langle\delta\mathrm{p},\mathrm{q},c[\mathrm{p}]\rangle = f$. We compute:

$$\begin{aligned}
ev\langle\delta\mathrm{p},\mathrm{q},c[\mathrm{p}]\rangle &= \langle\mathrm{p},\mathrm{ap}(\mathrm{q},\mathrm{q}[\mathrm{p}])\rangle\langle\delta\mathrm{p},\mathrm{q},c[\mathrm{p}]\rangle \\
&= \langle\delta\mathrm{p},\mathrm{q},\mathrm{ap}(c[\mathrm{p}],\mathrm{q})\rangle
\end{aligned}$$

Therefore it follows that $\mathrm{ap}(c[\mathrm{p}],\mathrm{q}) = \mathrm{q}[f] = b$. By $\eta$-expansion on $\Pi$-types, it follows that $\lambda(b) = \lambda(\mathrm{ap}(c[\mathrm{p}],\mathrm{q})) = c$. □

The next step is to observe that if $(F,\sigma)$ is a pseudo cwf-morphism between two cwfs supporting $\Pi$-types, then the definition of $(F,\sigma)$ preserving $\Pi$-types exactly amounts to $F$ preserving chosen dependent products.

**Lemma 2.** Let $(\mathbb{C},T)$ and $(\mathbb{C}',T')$ be cwfs supporting $\Pi$-types, and $(F,\sigma)$ be a pseudo cwf-morphism from $(\mathbb{C},T)$ to $(\mathbb{C}',T')$. Then $(F,\sigma)$ preserves $\Pi$-types if and only if the image of any chosen dependent product diagram is a dependent product diagram.

*Proof.* Through intricate calculations, we show that the conditions on $(F, \sigma)$ for preservation of Π-types exactly amount to preservation of chosen dependent product diagrams. See the appendix for details. □

Now, we can characterise preservation of Π-types for a pseudo cwf-morphism $(F, \sigma)$ as preservation of dependent products for $F$. As for preservation of finite limits, we use the inverse image construction to build a chosen dependent product diagram isomorphic to any given dependent product diagram.

**Proposition 12.** Let $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ be two cwfs supporting democracy, Π-types, Σ-types and identity types, and let $(F, \sigma)$ be a pseudo cwf-morphism preserving democracy. Then $(F, \sigma)$ preserves Π-types if and only if $F$ preserves dependent products.

*Proof.* First, we prove that since $(\mathbb{C}, T)$ supports Σ-types and identity types, preserving dependent products amounts to preserving the chosen dependent products along projections as generated by Π-types in the cwf structure. Indeed, take an arbitrary dependent product diagram:

$$\begin{tikzcd}
 & P \arrow[r] \arrow[d] \arrow[dl, bend right, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \arrow[d, "\Pi_f(g)"] \\
\Omega \arrow[r, "g"] & \Delta \arrow[r, "f"] & \Gamma
\end{tikzcd}$$

By the inverse image construction of Section 2.7, there is a unique isomorphism between this diagram and the following chosen dependent product diagram, where $\xi_f : \mathrm{p}_{\mathrm{Inv}(f)} \to f$ is the isomorphism in $\mathbb{C}/\Gamma$ described in Section 2.7.

$$\begin{tikzcd}
 & \Gamma\cdot\mathrm{Inv}(f)\cdot\Pi(\mathrm{Inv}(f), \mathrm{Inv}(g)[\xi_f])[\mathrm{p}] \arrow[r, "\langle \mathrm{pp},\mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"] \arrow[dl, bend right, "ev"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma\cdot\Pi(\mathrm{Inv}(f), \mathrm{Inv}(g)[\xi_f]) \arrow[d, "\mathrm{p}"] \\
\Gamma\cdot\mathrm{Inv}(f)\cdot\mathrm{Inv}(g)[\xi_f] \arrow[r, "\mathrm{p}"] & \Gamma\cdot\mathrm{Inv}(f) \arrow[r, "\mathrm{p}"] & \Gamma
\end{tikzcd}$$

Therefore preserving arbitrary dependent products amount to preserving the chosen dependent products along projections. The proposition follows by Lemma 2. □

When defining preservation of Π-types for pseudo cwf-morphisms, we postponed proving that it was stable under composition. Now it follows immediately from Proposition 12, since preserving dependent products is stable under composition. Likewise, we mentioned in Section 3.2.3 that from preservation of Π-types, abstraction was automatically preserved. It follows as well from Proposition 12, by the universal property of the dependent product diagram.

## 5. Rebuilding Types and Terms

We now construct pseudofunctors in the opposite direction. Following a method due to Bénabou and extended to type theory by Hofmann we construct a democratic cwf which supports extensional identity types and Σ-types from a category with finite limits. If we start with an lccc the resulting cwf also supports Π-types. To get a pseudofunctor we need to extend this construction to operate also on functors and natural transformations.

**Proposition 13.** There are pseudofunctors

$$\begin{aligned} H &: \mathbf{FL} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}} \\ H &: \mathbf{LCC} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}} \end{aligned}$$

defined by

$$\begin{aligned} H\mathbb{C} &= (\mathbb{C}, T_{\mathbb{C}}) \\ HF &= (F, \sigma_F) \\ H\phi &= (\phi, \psi_\phi) \end{aligned}$$

on 0-cells, 1-cells, and 2-cells, respectively, where $T_{\mathbb{C}}, \sigma_F$, and $\psi_\phi$ are defined in the following three subsections.

*Proof.* The remainder of this Section contains the proof. We will in turn show the action on 0-cells, 1-cells, 2-cells, and then prove pseudofunctoriality of $H$. □

### 5.1. *Action on 0-Cells*

As explained before, in categorical semantics of dependent types (going back to (Cartmell, 1986)) a type-in-context $A \in \mathrm{Type}(\Gamma)$ is represented by a *display map*, that is, as an object $\mathrm{p}_{\Gamma,A}$ in $\mathbb{C}/\Gamma$. A term in $\Gamma \vdash A$ is represented as a section of the display map for $A$, that is, a morphism $a$ such that $\mathrm{p}_A \circ a = \mathrm{id}_\Gamma$. Substitution in types is represented by pullback. This is essentially the technique used by Seely for interpreting Martin-Löf type theory in lcccs. However, as we already mentioned, it leads to a coherence problem.

To solve this problem (Hofmann, 1994) used a construction due to (Bénabou, 1985), which from any fibration builds an equivalent *split* fibration. In this way Hofmann built a category with attributes (cwa) from a locally cartesian closed category. He then showed that this cwa supports $\Pi, \Sigma$, and extensional identity types. This technique essentially amounts to associating to a type $A$, not only a display map, but a whole family of display maps, one for each substitution instance $A[\delta]$. In other words, we choose a pullback square for every possible substitution. This choice is split and hence solves the coherence problem. As we shall explain below this family takes the form of a functor, and we refer to it as a *functorial family.*

Here we reformulate Hofmann's construction using cwfs. See (Dybjer, 1996) for the correspondence between cwfs and cwas.

**Proposition 14.** Let $\mathbb{C}$ be a category with terminal object. Then we can build a democratic cwf $(\mathbb{C}, T_{\mathbb{C}})$ which supports Σ-types. If $\mathbb{C}$ has finite limits, then $(\mathbb{C}, T_{\mathbb{C}})$ also supports extensional identity types. If $\mathbb{C}$ is locally cartesian closed, then $(\mathbb{C}, T_{\mathbb{C}})$ also supports Π-types.

*Proof.* A *type* in $\mathrm{Type}_{\mathbb{C}}(\Gamma)$ is a *functorial family*, that is, a functor $\overrightarrow{A} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$

such that $\mathrm{cod} \circ \overrightarrow{A} = \mathrm{dom}$ and if $\begin{tikzcd}[column sep=small] \Omega \arrow[rr, "\alpha"] \arrow[dr, "\delta\alpha"'] & & \Delta \arrow[dl, "\delta"] \\ & \Gamma & \end{tikzcd}$ is a morphism in $\mathbb{C}/\Gamma$, then $\overrightarrow{A}(\alpha)$ is a pullback square:

$$\begin{tikzcd} \cdot \arrow[r, "\overrightarrow{A}(\delta,\alpha)"] \arrow[d, "\overrightarrow{A}(\delta\alpha)"'] & \cdot \arrow[d, "\overrightarrow{A}(\delta)"] \\ \Omega \arrow[r, "\alpha"'] & \Delta \end{tikzcd}$$

Following Hofmann, we denote the upper arrow of the square by $\overrightarrow{A}(\delta, \alpha)$.

A *term* $a : \Gamma \vdash \overrightarrow{A}$ is a section of $\overrightarrow{A}(\mathrm{id}_\Gamma)$, that is, a morphism $a : \Gamma \to \Gamma \cdot \overrightarrow{A}$ such that $\overrightarrow{A}(\mathrm{id}_\Gamma)a = \mathrm{id}_\Gamma$, where we have defined context extension by $\Gamma \cdot \overrightarrow{A} = \mathrm{dom}(\overrightarrow{A}(\mathrm{id}_\Gamma))$.

*Substitution in types.* Let $\gamma : \Delta \to \Gamma$ in $\mathbb{C}$ and $\overrightarrow{A} \in \mathrm{Type}(\Gamma)$. We define $\overrightarrow{A}[\gamma] \in \mathrm{Type}(\Delta)$ as follows.

$$\begin{aligned} \overrightarrow{A}[\gamma](\delta) &= \overrightarrow{A}(\gamma\delta) \\ \overrightarrow{A}[\gamma](\delta, \alpha) &= \overrightarrow{A}(\gamma\delta, \alpha) \end{aligned}$$

where $\delta : \Omega \to \Delta$ and $\alpha : \Xi \to \Omega$. It is easily verified that $\overrightarrow{A}[\gamma]$ satisfies the two conditions for types.

*Substitution in terms.* Let $\delta : \Delta \to \Gamma$, and $a : \Gamma \vdash \overrightarrow{A}$, that is, $a : \Gamma \to \Gamma \cdot \overrightarrow{A}$ such that $\overrightarrow{A}(\mathrm{id}_\Gamma) \circ a = \mathrm{id}_\Gamma$. Then $a[\delta]$ is defined as the unique mediating arrow in the following diagram:

$$\begin{tikzcd} \Delta \arrow[dr, dotted, "{a[\delta]}"] \arrow[drr, bend left, "a \circ \delta"] \arrow[ddr, bend right, "\mathrm{id}_\Delta"'] & & \\ & \Delta \circ \overrightarrow{A}[\delta] \arrow[r, "\overrightarrow{A}(\mathrm{id}_\Gamma, \delta)"] \arrow[d, "\overrightarrow{A}[\delta](\mathrm{id}_\Delta)"] & \Gamma \cdot \overrightarrow{A} \arrow[d, "\overrightarrow{A}(\mathrm{id}_\Gamma)"] \\ & \Delta \arrow[r, "\delta"'] & \Gamma \end{tikzcd}$$

It is a term of type $\overrightarrow{A}[\delta]$ by commutativity of the lower left triangle.

*Functoriality.* Since substitution in types is defined by composition, the cwf-laws follow immediately. Functoriality follows from the split choice of pullbacks of $\overrightarrow{A}$. Putting all this together, we now have built a functor $T_{\mathbb{C}} : \mathbb{C}^{op} \to \mathbf{Fam}$.

*Context comprehension.* Let $\Gamma \in \mathbb{C}$, and $\overrightarrow{A} \in \mathrm{Type}(\Gamma)$. As mentioned above, we define:

$$\Gamma \cdot \overrightarrow{A} = dom(\overrightarrow{A}(\mathrm{id}_\Gamma))$$

The first projection is $\mathrm{p}_{\overrightarrow{A}} = \overrightarrow{A}(\mathrm{id}_\Gamma) : \Gamma\cdot\overrightarrow{A} \to \Gamma$. The second projection $\mathrm{q}_{\overrightarrow{A}}$ is defined as the unique mediating arrow of the following pullback diagram:

$$\begin{tikzcd}
\Gamma\cdot\overrightarrow{A} \arrow[drr, dotted, "\mathrm{q}_{\overrightarrow{A}}"] \arrow[drrr, bend left, "\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}"] \arrow[ddrr, bend right, "\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}"'] & & & \\
& & \Gamma\cdot\overrightarrow{A}\cdot\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}] \arrow[r, "\overrightarrow{A}(\mathrm{id}_\Gamma{,}\mathrm{p}_{\overrightarrow{A}})"] \arrow[d, "\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](\mathrm{id}_{\Gamma\cdot\overrightarrow{A}})"] & \Gamma\cdot\overrightarrow{A} \arrow[d, "\overrightarrow{A}(\mathrm{id}_\Gamma)"] \\
& & \Gamma\cdot\overrightarrow{A} \arrow[r, "\mathrm{p}_{\overrightarrow{A}}"'] & \Gamma
\end{tikzcd}$$

Suppose now we have $\delta : \Delta \to \Gamma$ and $a : \Delta \vdash \overrightarrow{A}[\delta]$. By definition of terms we have in fact $a : \Delta \to \Delta\cdot\overrightarrow{A}[\delta]$. We define:

$$\langle \delta, a\rangle = \overrightarrow{A}(id_\Gamma, \delta) \circ a : \Delta \to \Gamma\cdot\overrightarrow{A}$$

It can be checked that for all $\overrightarrow{A} \in \mathrm{Type}(\Gamma)$, $\delta : \Delta \to \Gamma$ and $a : \Delta \vdash \overrightarrow{A}[\delta]$ we have $\mathrm{q}_{\overrightarrow{A}} \circ \langle\delta, a\rangle = \langle\langle\delta, a\rangle, a\rangle$, from which it follows easily that the cwf-laws for context comprehension are satisfied.

*Democracy.* The cwf $(\mathbb{C}, T_{\mathbb{C}})$ is democratic, since a context $\Gamma$ is represented by a functorial family with $\langle\rangle : \Gamma \to []$ as display map. We can easily build such a functorial family by $\overline{\Gamma} = \widehat{\langle\rangle} \in \mathrm{Type}([])$. We then have $[]\cdot\overline{\Gamma} = dom(\widehat{\langle\rangle}(\mathrm{id})) = \Gamma$. Thus the isomorphism between them is trivial.

$\Sigma$-*types.* Let $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma\cdot A)$. For $s : \Delta \to \Gamma$, the image of $\Sigma(A, B)$ is given by composing the images of $A$ and $B$. More formally, we define:

$$\begin{array}{rcl}
\Sigma(A, B)(s) & = & A(s) \circ B(A(\mathrm{id}, s)) \\
\Sigma(A, B)(s, \alpha) & = & B(A(\mathrm{id}, s), A(s, \alpha))
\end{array}$$

The construction of the corresponding pullback square can be illustrated by the following diagram. Intuitively, the chosen pullbacks for $\Sigma(A, B)$ are directly obtained by composing the chosen pullbacks for $A$ and for $B$.

$$\begin{tikzcd}
\cdot \arrow[r, "B(A(\mathrm{id}{,}s){,}A(s{,}\alpha))"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r, "B(\mathrm{id}{,}A(\mathrm{id}{,}s))"] \arrow[d, "B(A(\mathrm{id}{,}s))"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma\cdot A\cdot B \arrow[d, "B(\mathrm{id})"] \\
\cdot \arrow[r, "A(s{,}\alpha)"] \arrow[d, "A(s\alpha)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r, "A(\mathrm{id}{,}s)"] \arrow[d, "A(s)"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma\cdot A \arrow[d, "A(\mathrm{id})"] \\
\cdot \arrow[r, "\alpha"'] & B \arrow[r, "s"'] & \Gamma
\end{tikzcd}$$

It is easy to check that this defines a functor $\Sigma(A, B) : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ and that the necessary equations are satisfied so that we get a type $\Sigma(A, B) \in \mathrm{Type}(\Gamma)$ which satisfies the corresponding introduction rules, elimination rules and equations.

*Extensional identity types.* To build identity types, we require that the base category has finite limits. Let $\Gamma \in \mathbb{C}$, $A \in \mathrm{Type}(\Gamma)$, and $a, a' : \Gamma \vdash A$. If $s : \Delta \to \Gamma$, we define $\mathrm{I}_A(a,a')(s)$ as the equalizer of $a[s]$ and $a'[s]$ (seen as morphisms $\Delta \to \Delta \cdot A[s]$). If $\begin{tikzcd} \Delta' \arrow[rr, "\delta"] \arrow[dr, "s'"'] & & \Delta \arrow[dl, "s"] \\ & \Gamma & \end{tikzcd}$ is a morphism in $\mathbb{C}/\Gamma$, we define $\mathrm{I}_A(a,a')(\delta)$ as the upper square in the following diagram:

$$\begin{tikzcd}
{} \arrow[rr, dotted, "\gamma"] \arrow[d, "\mathrm{I}_A(a,a')(s\delta)"] & & {} \arrow[d, "\mathrm{I}_A(a,a')(s)"] \\
\Delta' \arrow[rr, "\delta"] \arrow[d, shift right, "a[s\delta]"'] \arrow[d, shift left, "a'[s\delta]"] & & \Delta \arrow[d, shift right, "a[s]"'] \arrow[d, shift left, "a'[s]"] \\
\Delta' \cdot A[s\delta] \arrow[rr, "\langle \delta \mathrm{p}, \mathrm{q} \rangle"'] & & \Delta \cdot A[s]
\end{tikzcd}$$

where $\gamma$ is yet to be defined. For this purpose, and to prove that the obtained square is a pullback, we need the following:

**Lemma 3.** In the diagram above, if $f : \mathrm{dom}(f) \to \Delta'$, then $f$ equalizes $a[s\delta]$ and $a'[s\delta]$ iff $\delta f$ equalizes $a[s]$ and $a'[s]$.

*Proof.* Follows by equational reasoning, exploiting the fact that in this cwf, for any term $a : \Gamma \vdash A$, we have the rather surprising equality $a = \langle \mathrm{id}_\Gamma, a \rangle$ (since $\langle \mathrm{id}_\Gamma, a \rangle = A(\mathrm{id}_\Gamma, \mathrm{id}_\Gamma) \circ a = a$). $\square$

We use this lemma as follows. We know that $\mathrm{I}_A(a,a')(s\delta)$ equalizes $a[s\delta]$ and $a'[s\delta]$, thus $\delta \circ \mathrm{I}_A(a,a')(s\delta)$ equalizes $a[s]$ and $a'[s]$. Thus by the equalizer property, $\delta \circ \mathrm{I}_A(a,a')(s\delta)$ factors in a unique way through $\mathrm{I}_A(a,a')(s)$, and we define $\gamma$ to be the unique morphism. It follows directly from Lemma 3 that this is a pullback square. This construction is functorial: both conditions (for $\mathrm{id}_s$ and $\delta_1 \circ \delta_2$) follow immediately by uniqueness of the factorisation through the equalizer. Thus we have shown that $\mathrm{I}_A(a,a') \in \mathrm{Type}(\Gamma)$. Introduction and elimination rules, stability under substitution and extensionality all follow from standard properties of equalizers.

$\Pi$-*types.* If $\mathbb{C}$ is an lccc, then the cwf $H(\mathbb{C})$ supports $\Pi$-types. Let $\overrightarrow{A}$ be a functorial family over $\Gamma$ and $\overrightarrow{B}$ over $\Gamma \cdot \overrightarrow{A}$. Then the value of the family $\Pi(\overrightarrow{A}, \overrightarrow{B})$ at substitution $\delta : \Delta \to \Gamma$ is $\Pi_{\overrightarrow{A}(\delta)}(\overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta)))$, where $\Pi_f$ is the right adjoint of $f^*$ in an lccc. If $\alpha : \Omega \to \Delta$ and $\delta : \Delta \to \Gamma$, we define a morphism $\Pi(\overrightarrow{A}, \overrightarrow{B})(\delta, \alpha)$ yielding a pullback diagram. For this purpose, consider the following chain of isomorphisms in $\mathbb{C}/\Omega$:

$$\begin{aligned}
\Pi_{\overrightarrow{A}(\delta\alpha)} \overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta\alpha)) &= \Pi_{\overrightarrow{A}(\delta\alpha)} \overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta) \overrightarrow{A}(\delta, \alpha)) \\
&\cong \Pi_{\overrightarrow{A}(\delta\alpha)} (\overrightarrow{A}(\delta, \alpha))^* (\overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta))) \\
&\cong \alpha^* (\Pi_{\overrightarrow{A}(\delta)} \overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta)))
\end{aligned}$$

The first isomorphism is by uniqueness of the pullback of $\overrightarrow{B}(\mathrm{id}, \delta)$ along $\overrightarrow{A}(\delta, \alpha)$, while the second is by the Beck-Chevalley condition applied to the pullback square of $\overrightarrow{A}(\delta, \alpha)$.

Let us call this isomorphism $\phi$. The action of $\alpha^*$ also gives a canonical morphism $h$ : $\mathrm{dom}(\alpha^*(\Pi_{\overrightarrow{A}(\delta)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)))) \to \mathrm{dom}(\Pi_{\overrightarrow{A}(\delta)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)))$, thus we define:

$$\Pi(\overrightarrow{A},\overrightarrow{B})(\delta,\alpha) = h\phi : \mathrm{dom}(\Pi(\overrightarrow{A},\overrightarrow{B})(\delta)) \to \mathrm{dom}(\Pi(\overrightarrow{A},\overrightarrow{B})(\delta\alpha))$$

This defines a pullback square since it is obtained from an isomorphism and a pullback. Hence the definition of the functorial family $\Pi(\overrightarrow{A},\overrightarrow{B})$ is complete, since the equations come from the universal property of the pullback. The fact that $\Pi(\overrightarrow{A},\overrightarrow{B})[\delta]$ and $\Pi(\overrightarrow{A}[\delta],\overrightarrow{B}[\langle\delta\mathrm{p},\mathrm{q}\rangle])$ coincide on objects (of $\mathbb{C}/\Gamma$) is a straightforward calculation, from which the fact that they coincide on morphisms can be directly deduced.

The combinators $\lambda$ and ap come from natural applications of the adjunction $(\overrightarrow{A}(\mathrm{id}))^* \dashv \Pi_{\overrightarrow{A}(\mathrm{id})}$, and the computation rules follow from the properties of adjunctions. As in (Hofmann, 1994), the behaviour of the combinators $\lambda$ and ap under substitution is obtained by reworking the proof of the Beck-Chevalley conditions for lcccs.

□

### 5.2. *Action on 1-Cells*

Suppose that $\mathbb{C}$ and $\mathbb{C}'$ have finite limits and that $F : \mathbb{C} \to \mathbb{C}'$ preserves them. As described in the previous section, $\mathbb{C}$ and $\mathbb{C}'$ give rise to cwfs $(\mathbb{C}, T_{\mathbb{C}})$ and $(\mathbb{C}', T_{\mathbb{C}'})$. In order to extend $F$ to a pseudo cwf-morphism, we need to define, for each object $\Gamma$ in $\mathbb{C}$, a **Fam**-morphism $(\sigma_F)_\Gamma : T_{\mathbb{C}}(\Gamma) \to T_{\mathbb{C}'}F(\Gamma)$. Unfortunately, unless $F$ is full, it does not seem possible to embed faithfully a functorial family $\overrightarrow{A} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ into a functorial family over $F\Gamma$ in $\mathbb{C}'$. However, there is such an embedding for display maps (just apply $F$) from which we freely regenerate a functorial family from the obtained display map.

*The "hat" construction.* As remarked by Hofmann, any morphism $f : \Delta \to \Gamma$ in a category $\mathbb{C}$ with a (not necessarily split) choice of finite limits generates a functorial family $\widehat{f} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$. If $\delta : \Delta \to \Gamma$ then $\widehat{f}(\delta) = \delta^*(f)$, where $\delta^*(f)$ is obtained by taking the pullback of $f$ along $\delta$ ($\delta^*$ is known as the *pullback functor*):

$$\begin{tikzcd} {} \arrow[r] \arrow[d, "\delta^*(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & {} \arrow[d, "f"] \\ \Delta \arrow[r, "\delta"'] & \Gamma \end{tikzcd}$$

Note that we can always choose pullbacks such that $\widehat{f}(\mathrm{id}_\Gamma) = \mathrm{id}_\Gamma^*(f) = f$. If $\begin{tikzcd}[column sep=small] \Omega \arrow[rr, "\alpha"] \arrow[dr, "\delta\alpha"'] & & \Delta \arrow[dl, "\delta"] \\ & \Gamma & \end{tikzcd}$ is a morphism in $\mathbb{C}/\Gamma$, we define $\widehat{f}(\alpha)$ as the left square in the following diagram:

$$\begin{tikzcd} {} \arrow[r, "\widehat{f}(\delta,\alpha)"] \arrow[d, "\widehat{f}(\delta\alpha)"'] \arrow[dr, phantom, "\lrcorner", very near start] & {} \arrow[r] \arrow[d, "\widehat{f}(\delta)"] \arrow[dr, phantom, "\lrcorner", very near start] & {} \arrow[d, "f"] \\ \Delta' \arrow[r, "\alpha"'] & \Delta \arrow[r, "\delta"'] & \Gamma \end{tikzcd}$$

This is a pullback, since both the outer square and the right square are pullbacks.

*Translation of types.* The hat construction can be used to extend $F$ to types:

$$\sigma_F(\overrightarrow{A}) = \widehat{F(\overrightarrow{A}(\mathrm{id}))}$$

Note that $F(\Gamma{\cdot}\overrightarrow{A}) = F(\mathrm{dom}(\overrightarrow{A}(\mathrm{id}))) = \mathrm{dom}(F(\overrightarrow{A}(\mathrm{id}))) = \mathrm{dom}(\sigma_\Gamma(\overrightarrow{A})(\mathrm{id})) = F\Gamma{\cdot}\sigma_\Gamma(\overrightarrow{A})$, so context comprehension is preserved on the nose. However, substitution on types is *not* preserved on the nose. Hence we have to define a coherent family of isomorphisms $\theta_{\overrightarrow{A},\delta}$.

*Completion of cwf-morphisms.* Fortunately, whenever $F$ preserves finite limits there is a canonical way to generate all the remaining data.

**Lemma 4 (Generation of isomorphisms).** Let $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ be two cwfs, $F : \mathbb{C} \to \mathbb{C}'$ a functor preserving finite limits, $\sigma_\Gamma : \mathrm{Type}(\Gamma) \to \mathrm{Type}'(F\Gamma)$ a family of functions, and $\rho_{\Gamma,A} : F(\Gamma \cdot A) \to F\Gamma \cdot \sigma_\Gamma(A)$ a family of isomorphisms such that $\mathrm{p}\rho_{\Gamma,A} = F\mathrm{p}$. Then there exists a unique choice of functions $\sigma_\Gamma^A$ on terms and of isomorphisms $\theta_{A,\delta}$ such that $(F, \sigma)$ is a pseudo cwf-morphism.

*Proof.* By item *(2)* of Proposition 5, the unique way to extend $\sigma$ to terms is to set $\sigma_\Gamma^A(a) = \mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]$. To generate $\theta$, we use the two squares below:

$$\begin{tikzcd}
F\Delta\cdot\sigma_\Gamma(A)[F\delta] \arrow[r, "\langle (F\delta)\mathrm{p},\mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & F\Gamma\cdot\sigma_\Gamma(A) \arrow[d, "\mathrm{p}"] \\
F\Delta \arrow[r, "F\delta"'] & F\Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
F\Delta\cdot\sigma_\Delta(A[\delta]) \arrow[r, "\rho_{\Gamma,A}F(\langle \delta\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}"] \arrow[d, "\mathrm{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & F\Gamma\cdot\sigma_\Gamma(A) \arrow[d, "\mathrm{p}"] \\
F\Delta \arrow[r, "F\delta"'] & F\Gamma
\end{tikzcd}$$

The first square is a substitution pullback. The second is a pullback because $F$ preserves finite limits and $\rho_{\Gamma,A}$ and $\rho_{\Delta,A[\delta]}$ are isomorphisms. The isomorphism $\theta_{A,\delta}$ is defined as the unique mediating morphism from the first to the second. It follows from the universal property of pullbacks that the family $\theta$ satisfies the necessary naturality and coherence conditions. There is no other choice for $\theta_{A,\delta}$, because if $(F, \sigma)$ is a pseudo cwf-morphism with families of isomorphisms $\theta$ and $\rho$, then $\rho_{\Gamma,A}F(\langle \delta\mathrm{p}, \mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}\theta_{A,\delta} = \langle (F\delta)\mathrm{p}, \mathrm{q}\rangle$. Hence if $F$ preserves finite limits, $\theta_{A,\delta}$ must coincide with the mediating morphism. □

*Preservation of additional structure.* As a pseudo cwf-morphism, $(F, \sigma_F)$ automatically preserves $\Sigma$-types.

Since the democratic structure of $(\mathbb{C}, T_\mathbb{C})$ and $(\mathbb{C}', T_{\mathbb{C}'})$ is trivial it is easy to prove that it is preserved by $(F, \sigma_F)$:

**Proposition 15.** If $F : \mathbb{C} \to \mathbb{C}'$ preserves finite limits, then $\sigma_F$ preserves democracy.

*Proof.* The functor $F$ preserves finite limits and thus preserves the terminal object. Let $\iota : [] \to F[]$ denote the inverse to the terminal projection. Note that since the two involved cwfs have been built with Hofmann's construction, their democratic structure is trivial; we have $[]\cdot\overline{\Gamma} = \Gamma$ and $\gamma_\Gamma = \mathrm{id}$. In particular, we have $F([]\cdot\overline{\Gamma}) = F(\Gamma) = []\cdot\overline{F\Gamma}$. Thus to get preservation of the democratic structure, it is natural to choose:

$$d_\Gamma = \langle \iota, \mathrm{q}\rangle \rho^{-1}_{[],\overline{\Gamma}} : []\cdot\sigma_{[]}(\overline{\Gamma}) \to []\cdot\overline{F\Gamma}[\langle\rangle]$$

which makes the coherence condition essentially trivial. □

All the other type constructors are preserved: for Σ-types it is automatic, for identity types and Π-types it follows from Propositions 10 and 12.

### 5.3. *Action on* 2-*Cells*

In a similar way as for 1-cells, we shall show that under certain conditions a natural transformation $\phi : F \stackrel{\bullet}{\to} G$, where $(F, \sigma)$ and $(G, \tau)$ are pseudo cwf-morphisms, can be completed to a pseudo cwf-transformation $(\phi, \psi_\phi)$.

**Lemma 5 (Completion of pseudo cwf-transformations).** Suppose $(F, \sigma)$ and $(G, \tau)$ are pseudo cwf-morphisms from $(\mathbb{C}, T)$ to $(\mathbb{C}', T)$ such that $F$ and $G$ preserve finite limits and $\phi : F \stackrel{\bullet}{\to} G$ is a natural transformation, then there exists a family of morphisms $(\psi_\phi)_{\Gamma,A} : \sigma_\Gamma(A) \to \tau_\Gamma(A)[\phi_\Gamma]$ such that $(\phi, \psi_\phi)$ is a pseudo cwf-transformation from $(F, \sigma)$ to $(G, \tau)$.

*Proof.* We set $\psi_{\Gamma,A} = \langle \mathrm{p}, \mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle : F\Gamma\cdot\sigma_\Gamma A \to F\Gamma\cdot\tau_\Gamma(A)[\phi_\Gamma]$. The coherence law follows from the universal property of a well-chosen pullback square, see the appendix for details. □

This completion operation on 2-cells commutes with units and both notions of composition, as will be crucial to prove pseudofunctoriality of $H$:

**Lemma 6.** Completion of pseudo cwf-transformations commutes with both notions of composition. More precisely, if $\phi : F \stackrel{\bullet}{\to} G$ and $\phi' : G \stackrel{\bullet}{\to} H$, then

$$(\phi', \psi_{\phi'}) \bullet (\phi, \psi_\phi) = (\phi' \bullet \phi, \psi_{\phi' \bullet \phi})$$

Likewise if $\phi : F \stackrel{\bullet}{\to} G$ and $\phi' : F' \to G'$,

$$(\phi', \psi_{\phi'})(\phi, \psi_\phi) = (\phi'\phi, \psi_{\phi'\phi})$$

Finally, for all pseudo cwf-morphism $(F, \sigma)$ we have $1_{(F,\sigma)} = (1_F, \psi_{1_F})$.

*Proof.* The first equality is a straightforward verification. The second requires a more involved calculation similar to the one used to prove Lemma 5, see the appendix. Finally, the third equality follows from the remark that by definition of $\psi_{1_F}$, we have $(\psi_{1_F})_{\Gamma,A} = \mathrm{id}_{\Gamma\sigma_\Gamma A}$ for all Γ, $A$. □

### 5.4. *Pseudofunctoriality of H*

First note that $H$ is *not* a functor, because for $F : \mathbb{C} \to \mathbb{D}$ with finite limits and functorial family $\overrightarrow{A}$ over Γ (in $\mathbb{C}$), $\sigma_\Gamma(\overrightarrow{A})$ forgets all information on $\overrightarrow{A}$ except its display map $\overrightarrow{A}(\mathrm{id})$, and later extends $F(\overrightarrow{A}(\mathrm{id}))$ to an independent functorial family.

However, we shall prove:

**Proposition 16.** $H : \mathbf{FL} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ and $H : \mathbf{LCC} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ are pseudofunctors.

*Proof.* First, note that as proved in Lemma 6, $H$ is functorial on 2-cells.

For each $\mathbb{C}$ we need an invertible 2-cell $H_{\mathbb{C}} : Id_{(\mathbb{C},T_{\mathbb{C}})} \to H(Id_{\mathbb{C}})$, this will be the identity 2-cell since we have in fact $H(Id_{\mathbb{C}}) = (Id_{\mathbb{C}}, \sigma_{Id_{\mathbb{C}}}) = Id_{(\mathbb{C},T_{\mathbb{C}})}$ by construction of $\sigma_{Id_{\mathbb{C}}}$.

For each two functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{E}$ we need an isomorphism $H_{F,G} : HG \circ HF \to H(G \circ F)$, natural in $F$ and $G$. It is given by $H_{F,G} = (1_{GF}, \psi_{1_{GF}})$. The naturality condition amounts to the commutativity of the following square:

$$\begin{tikzcd}
(G,\sigma_G)(F,\sigma_F) \arrow[r, "(1_{GF},\psi_{1_{GF}})"] \arrow[d, "(\phi,\psi_\phi)(\phi',\psi_{\phi'})"] & (GF,\sigma_{GF}) \arrow[d, "(\phi'\phi,\psi_{\phi'\phi})"] \\
(G',\sigma_{G'})(F',\sigma_{F'}) \arrow[r, "(1_{GF},\psi_{1_{GF}})"] & (G'F',\sigma_{G'F'})
\end{tikzcd}$$

This is a direct consequence of Lemma 6. The coherence laws with respect to associativity of composition and identities also follow from Lemma 6. In fact, Lemma 5 implies that to check the validity of any equation involving vertical and horizontal compositions of pseudo cwf-transformations built with Lemma 5 and identity pseudo cwf-transformations, it suffices to check the equality of the corresponding base natural transformation, ignoring the modifications. □

## 6. The Biequivalences

**Theorem 1.** We have the following biequivalences of 2-categories.

$$\mathbf{FL} \underset{U}{\overset{H}{\rightleftarrows}} \mathbf{CwF}^{\mathrm{I}_{\mathrm{ext}}\Sigma}_{\mathrm{dem}} \qquad\qquad \mathbf{LCC} \underset{U}{\overset{H}{\rightleftarrows}} \mathbf{CwF}^{\mathrm{I}_{\mathrm{ext}}\Sigma\Pi}_{\mathrm{dem}}$$

*Proof.* Since $UH = \mathrm{Id}$ (the identity 2-functor) it suffices to construct pseudonatural transformations of pseudofunctors:

$$\mathrm{Id} \underset{\epsilon}{\overset{\eta}{\rightleftarrows}} HU$$

which are inverse up to invertible modifications. Since $HU(\mathbb{C},T) = (\mathbb{C},T^{\mathbb{C}})$, these pseudonatural transformations are families of equivalences of cwfs:

$$(\mathbb{C},T) \underset{\epsilon_{(\mathbb{C},T)}}{\overset{\eta_{(\mathbb{C},T)}}{\rightleftarrows}} (\mathbb{C},T^{\mathbb{C}})$$

which satisfy the required conditions for pseudonatural transformations.

*Construction of* $\eta_{(\mathbb{C},T)}$. Using Lemma 4, we just need to define a base functor, which will be $\mathrm{Id}_{\mathbb{C}}$, and a family $\sigma^\eta_\Gamma$ which translates types (in the sense of $T$) to functorial families. This is easy, since types in the cwf $(\mathbb{C},T)$ come equipped with a chosen behaviour under substitution. Given $A \in \mathrm{Type}(\Gamma)$, we define:

$$\begin{array}{rcl}
\sigma^\eta_\Gamma(A)(\delta) & = & \mathrm{p}_{A[\delta]} \\
\sigma^\eta_\Gamma(A)(\delta,\gamma) & = & \langle \gamma\mathrm{p}, \mathrm{q}\rangle
\end{array}$$

Preservation of type constructors follows from Propositions 10 and 12 and the fact that the identity functor preserves finite limits and dependent products.

For each pseudo cwf-morphism $(F, \sigma)$, the pseudonaturality square relates two pseudo cwf-morphisms whose base functor is $F$. Hence, the necessary invertible pseudo cwf-transformation is obtained using Lemma 5 from the identity natural transformation on $F$. The coherence conditions are straightforward consequences of Lemma 5.

*Construction of* $\epsilon_{(\mathbb{C},T)}$. As for $\eta$, the base functor for $\epsilon_{(\mathbb{C},T)}$ is $\mathrm{Id}_{\mathbb{C}}$. Using Lemma 4 again we need, for each context $\Gamma$, a function $\sigma^{\epsilon}_{\Gamma}$ which given a functorial family $\overrightarrow{A}$ over $\Gamma$ will build a syntactic type $\sigma^{\epsilon}_{\Gamma}(\overrightarrow{A}) \in \mathrm{Type}(\Gamma)$. In other words, we need to find a syntactic representative of an arbitrary display map, that is, an arbitrary morphism in $\mathbb{C}$. We use the inverse image:

$$\sigma^{\epsilon}_{\Gamma}(\overrightarrow{A}) = \mathrm{Inv}(\overrightarrow{A}(\mathrm{id})) \in \mathrm{Type}(\Gamma)$$

As for $\eta_{(\mathbb{C},T)}$, type constructors are preserved by Propositions 10 and 12. The family $\epsilon$ is pseudonatural for the same reason as $\eta$ above.

*Invertible modifications.* For each cwf $(\mathbb{C}, T)$, we need to define invertible pseudo cwf-transformations $m_{(\mathbb{C},T)} : (\epsilon\eta)_{(\mathbb{C},T)} \to \mathrm{id}_{(\mathbb{C},T)}$ and $m'_{(\mathbb{C},T)} : (\eta\epsilon)_{(\mathbb{C},T)} \to \mathrm{id}_{(\mathbb{C},T)}$. As pseudo cwf-transformations between pseudo cwf-morphisms with the same base functor, their first component will be the identity natural transformation, and the second will be generated by Lemma 5. The coherence law for modifications is a consequence of Lemma 5. □

## 7. Conclusion

The cwf morphism $\eta_{(\mathbb{C},T)}$ describes the *interpretation* of the cwf $(\mathbb{C}, T)$ into the cwf $(\mathbb{C}, T^{\mathbb{C}})$ obtained by the Bénabou-Hofmann construction. It is analogous to Hofmann's interpretation of a category with attributes in a lccc (Hofmann, 1994). Note that $\eta_{(\mathbb{C},T)}$ is a *strict* cwf morphism, although morphisms in the categories $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ in general are only required to be *pseudo* cwf-morphisms. The strictness of $\eta$ is important for an interpretation, since it means that the laws of cwfs are preserved strictly and not only up to isomorphism.

In order to prove our result we were forced to consider categories of cwfs with pseudo cwf morphisms. We were unable to prove a (bi)equivalence with categories of cwfs and strict cwf morphisms. For example, there is no obvious candidate for a strict replacement for $\epsilon_{(\mathbb{C},T)}$ since we must then construct a syntactic type over $\Gamma$ for each semantic type over $\Gamma$ (that is, an object of the slice category over $\Gamma$). The need to consider pseudo cwf morphisms indicates that the connection between Martin-Löf type theory and locally cartesian closed categories is not as tight as for example the connection between the simply typed lambda calculus and cartesian closed categories, or between syntactically defined Martin-Löf type theories and cwfs. So is Martin-Löf type theory with extensional identity types, $\Sigma$- and $\Pi$-types, an internal language for lcccs? Yes, it is an internal language "up to isomorphism".

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## Appendix A. Additional proofs about cwfs and pseudo cwf morphisms

**Lemma 7.** Let $(\mathbb{C}, T)$ be a cwf. Let $\delta : \Delta \to \Gamma$ be a substitution, $A \cong_f A'$ isomorphic types in Type$(\Gamma)$ and $a : \Gamma \vdash A$ be a term. Then:

$$(\{f\}(a))[\delta] = \{\boldsymbol{T}(\delta)(f)\}(a[\delta])$$

*Proof.* Follows from the following equational reasoning.

$$\begin{array}{rcl}(\{f\}(a))[\delta] & = & \mathrm{q}[f\langle \mathrm{id}, a\rangle\delta] \\ & = & \mathrm{q}[f\langle \delta, a[\delta]\rangle] \\ & = & \mathrm{q}[f\langle \delta\mathrm{p}, \mathrm{q}\rangle][\langle \mathrm{id}, a[\delta]\rangle] \\ & = & \{\langle \mathrm{p}, \mathrm{q}[f\langle \delta\mathrm{p}, \mathrm{q}\rangle]\rangle\}(a[\delta]) \\ & = & \{\boldsymbol{T}(\delta)(f)\}(a[\delta])\end{array}$$

□

**Lemma 8.** Let $(F, \sigma) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ be a pseudo cwf-morphism with families of isomorphisms $\theta$ and $\rho$. Then for any $\delta : \Delta \to \Gamma$ in $\mathbb{C}$ and type $A \in$ Type$(\Gamma)$, we have:

$$F(\langle \delta\mathrm{p}, \mathrm{q}\rangle) = \rho^{-1}_{\Gamma,A}\langle F(\delta)\mathrm{p}, \mathrm{q}\rangle\theta^{-1}_{A,\delta}\rho_{\Delta,A[\delta]}$$

*Proof.* Direct calculation.

$$\begin{array}{rcl}F(\langle \delta\mathrm{p}, \mathrm{q}\rangle) & =_1 & \rho^{-1}_{\Gamma,A}\langle F(\delta\mathrm{p}), \{\theta^{-1}_{A,\delta\mathrm{p}}\}(\sigma^{A[\delta\mathrm{p}]}_{\Delta\cdot A[\delta]}(\mathrm{q}))\rangle \\ & =_2 & \rho^{-1}_{\Gamma,A}\langle F(\delta\mathrm{p}), \{\theta^{-1}_{A,\delta\mathrm{p}}\}(\{\theta_{A[\delta],\mathrm{p}}\}(\mathrm{q}[\rho_{\Delta,A[\delta]}]))\rangle \\ & =_3 & \rho^{-1}_{\Gamma,A}\langle F(\delta\mathrm{p}), \{\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{A,\delta})\}(q[\rho_{\Delta,A[\delta]}])\rangle \\ & =_4 & \rho^{-1}_{\Gamma,A}\langle F(\delta\mathrm{p}), \mathrm{q}[\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{A,\delta})\langle \mathrm{id}, q[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\ & =_5 & \rho^{-1}_{\Gamma,A}\langle F(\delta\mathrm{p}), \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[\theta^{-1}_{A,\delta}\langle (F\mathrm{p})\mathrm{p}, \mathrm{q}\rangle]\rangle\langle \mathrm{id}, q[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\ & = & \rho^{-1}_{\Gamma,A}\langle F(\delta\mathrm{p}), \mathrm{q}[\theta^{-1}_{A,\delta}\langle F\mathrm{p}, \mathrm{q}[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\ & =_6 & \rho^{-1}_{\Gamma,A}\langle F(\delta)\mathrm{p}\rho_{\Delta,A[\delta]}, \mathrm{q}[\theta^{-1}_{A,\delta}\langle \mathrm{p}\rho_{\Delta,A[\delta]}, \mathrm{q}[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\ & = & \rho^{-1}_{\Gamma,A}\langle F(\delta)\mathrm{p}, \mathrm{q}[\theta^{-1}_{A,\delta}]\rangle\rho_{\Delta,A[\delta]} \\ & =_7 & \rho^{-1}_{\Gamma,A}\langle F(\delta)\mathrm{p}, \mathrm{q}\rangle\theta^{-1}_{A,\delta}\rho_{\Delta,A[\delta]}\end{array}$$

where (1) is by preservation of substitution extention (Proposition 4), (2) is by the preservation law (b) for the second projection, (3) is by coherence of $\theta$, (4) is by definition of the coercion $\{\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{A,\delta})\}$, (5) is by definition of $\boldsymbol{T'}$, (6) is by the preservation law (a) for the first projection. The other equations use basic manipulation of cwf combinators, with (7) using additionally that $\theta^{-1}_{A,\delta}$ is a morphism in $\boldsymbol{T'}(F\Delta)$, so $\mathrm{p}\theta^{-1}_{A,\delta} = \mathrm{p}$. □

**Lemma 9.** If $(F, \sigma)$ is a pseudo cwf-morphism from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ and $f : A \to B$ is a morphism in $\boldsymbol{T}(\Gamma)$, then the coercion $\{f\}$ commutes with $\sigma$ in the following way, for each $a : \Gamma \vdash A$:

$$\sigma_\Gamma^B(\{f\}(a)) = \{\boldsymbol{\sigma}_\Gamma(f)\}(\sigma_\Gamma^A(a))$$

*Proof.* Direct calculation.

$$\begin{array}{rcl}
\sigma_\Gamma^B(\{f\}(a)) & =_1 & \mathrm{q}[\rho_{\Gamma,B} F(\langle \mathrm{id}, \{f\}(a)\rangle)] \\
& =_2 & \mathrm{q}[\rho_{\Gamma,B} F(\langle \mathrm{id}, \mathrm{q}[f\langle \mathrm{id}, a\rangle]\rangle)] \\
& =_3 & \mathrm{q}[\rho_{\Gamma,B} F(f\langle \mathrm{id}, a\rangle)] \\
& =_4 & \mathrm{q}[\boldsymbol{\sigma}_\Gamma(f)\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)] \\
& =_3 & \mathrm{q}[\boldsymbol{\sigma}_\Gamma(f)\langle \mathrm{p}\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle), \mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]\rangle \\
& =_5 & \mathrm{q}[\boldsymbol{\sigma}_\Gamma(f)\langle \mathrm{id}, \mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]\rangle \\
& =_2 & \{\boldsymbol{\sigma}_\Gamma(f)\}(\mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]) \\
& =_1 & \{\boldsymbol{\sigma}_\Gamma(f)\}(\sigma_\Gamma^A(a))
\end{array}$$

Where (1) is by Proposition 5, (2) by definition of coercions, (3) by basic manipulation of cwf combinators (for the first instance, noting that $f$ is a morphism in $\boldsymbol{T}(\Gamma)$) (4) by definition of $\boldsymbol{\sigma}$ and (5) by the law (a) for the preservation of the first projection by $F$. □

## Appendix B. Additional proofs about the forgetful 2-functor $U$

**Lemma 10 (Propagation of isomorphisms).** Isomorphisms propagate through types in several different ways. Suppose that you have $A, A' \in \mathit{Type}(\Gamma)$, $B, B' \in \mathit{Type}(\Gamma \cdot A)$, then

(1) If $B \cong B'$, then $\Sigma(A, B) \cong \Sigma(A, B')$
(2) If $A \cong_f A'$, then $\Sigma(A, B) \cong \Sigma(A', B[f^{-1}])$
(3) If $A \cong_f A'$ and $a, a' \in \Gamma \vdash A$, then $\mathrm{I}_A(a, a') \cong \mathrm{I}_{A'}(\{f\}(a), \{f\}(a'))$

*Proof.* (1) is obvious, since $\Gamma \cdot \Sigma(A, B)$ is isomorphic to $\Gamma \cdot A \cdot B$. For (2), we give the following two isomorphisms:

$$\begin{array}{rcl}
\langle p, \mathrm{pair}(\mathrm{q}[f\langle \mathrm{p}, \pi_1(\mathrm{q})\rangle], \pi_2(\mathrm{q}))\rangle & : & \Gamma \cdot \Sigma(A, B) \to \Gamma \cdot \Sigma(A', B[f^{-1}]) \\
\langle p, \mathrm{pair}(\mathrm{q}[f^{-1}\langle \mathrm{p}, \pi_1(\mathrm{q})\rangle], \pi_2(\mathrm{q}))\rangle & : & \Gamma \cdot \Sigma(A', B[f^{-1}]) \to \Gamma \cdot \Sigma(A, B)
\end{array}$$

A simple calculation shows that they have the right types and that they are inverse of one another. It is obvious that they are isomorphisms of types. (3) is also obvious since by extensionality, $\langle \mathrm{p}, \mathrm{r}\rangle$ typechecks in both directions and is its own inverse. □

*Proof of Lemma 1.* Exploiting Lemma 10 and preservation of substitution on types and terms, a careful (but straightforward) calculation derives the following type isomorphism:

$$\sigma_\Gamma(\mathrm{Inv}(\delta)) \cong \Sigma(\overline{F\Delta}[\langle\rangle], \mathrm{I}_{\overline{F\Gamma}[\langle\rangle]}(C(\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma \cdot \overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle])), C(\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma \cdot \overline{\Delta}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}]))))$$

where $C(-)$ is an invertible context given by:

$$C(M) = \{\boldsymbol{T'}(\iota\langle\rangle)(d_\Gamma)\theta^{-1}_{\overline{\Gamma},\langle\rangle}\}(M)[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}\theta_{\overline{\Delta},\langle\rangle}\boldsymbol{T'}(\iota\langle\rangle)(d^{-1}_\Delta)]$$

Hence, it remains to show the following equalities:

$$\begin{array}{rcll}\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle,\mathrm{q}\rangle]) & = & C^{-1}(\overline{F\delta}[\langle\langle\rangle,\mathrm{q}\rangle]) & (1)\\ \sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma\mathrm{p}]) & = & C^{-1}(\mathrm{q}[\gamma_{F\Gamma}\mathrm{p}]) & (2)\end{array}$$

Let us focus on (1). Using preservation of substitution on terms, coherence of $\theta$ and the basic computation laws in cwfs, we derive:

$$\begin{array}{rcl}\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle,\mathrm{q}\rangle]) & = & \sigma^{\overline{\Gamma}[\mathrm{p}][\langle\langle\rangle,\mathrm{q}\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle,\mathrm{q}\rangle])\\ & =_1 & \{\theta_{\overline{\Gamma}[\mathrm{p}],\langle\langle\rangle,\mathrm{q}\rangle}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta})[F(\langle\langle\rangle,\mathrm{q}\rangle)])\\ & =_2 & \{\theta_{\overline{\Gamma},\langle\rangle}\}(\{\boldsymbol{T'}(F(\langle\langle\rangle,\mathrm{q}\rangle))(\theta^{-1}_{\overline{\Gamma},\mathrm{p}})\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta})[F(\langle\langle\rangle,\mathrm{q}\rangle)]))\\ & =_3 & \{\theta_{\overline{\Gamma},\langle\rangle}\}(\{\theta^{-1}_{\overline{\Gamma},\mathrm{p}}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta}))[F(\langle\langle\rangle,\mathrm{q}\rangle)])\end{array}$$

where (1) is by preservation of substitution on terms, (2) is by coherence of $\theta$ and (3) is by Lemma 7.

Let us now focus on $\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta})$, to see how terms created from substitution using democracy are transformed by the action of the cwf-morphism.

$$\begin{array}{rcl}\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta}) & =_1 & \sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\mathrm{q}[\gamma_\Gamma\delta\gamma^{-1}_\Delta])\\ & =_2 & \sigma^{\overline{\Gamma}[\mathrm{p}][\gamma_\Gamma\delta\gamma^{-1}_\Delta]}_{[]\cdot\overline{\Delta}}(\mathrm{q}[\gamma_\Gamma\delta\gamma^{-1}_\Delta])\\ & =_3 & \{\theta_{\overline{\Gamma}[\mathrm{p}],\gamma_\Gamma\delta\gamma^{-1}_\Delta}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Gamma}}(\mathrm{q})[F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)])\\ & =_4 & \{\theta_{\overline{\Gamma}[\mathrm{p}],\gamma_\Gamma\delta\gamma^{-1}_\Delta}\}(\{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}])[F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)])\\ & =_5 & \{\theta_{\overline{\Gamma}[\mathrm{p}],\gamma_\Gamma\delta\gamma^{-1}_\Delta}\}(\{\boldsymbol{T'}(F(\gamma_\Gamma\delta\gamma^{-1}_\Delta))(\theta_{\overline{\Gamma},\mathrm{p}})\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}][F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)]))\\ & =_6 & \{\theta_{\overline{\Gamma},\mathrm{p}\gamma_\Gamma\delta\gamma^{-1}_\Delta}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)])\\ & =_7 & \{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)])\end{array}$$

where (1) is by Definition 6, (2) is by uniqueness of the morphism $[]\cdot\overline{\Delta}\to[]$ (since $[]$ is terminal), (3) is by preservation of substitution on terms, (4) is by law (b) for the preservation of second projection, (5) is by Lemma 7, (6) is by coherence of $\theta$, and (7) is by uniqueness of the morphism $[]\cdot\overline{\Delta}\to[]$, since $[]$ is terminal.

Using this, we continue simplifying $\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle,\mathrm{q}\rangle])$:

$$\begin{array}{ll}
& \sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle,\mathrm{q}\rangle]) \\
= & \{\theta_{\overline{\Gamma},\langle\rangle}\}(\{\theta^{-1}_{\overline{\Gamma},\mathrm{p}}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\overline{\Delta}}(\overline{\delta}))[F(\langle\langle\rangle,\mathrm{q}\rangle)]) \\
=_1 & \{\theta_{\overline{\Gamma},\langle\rangle}\}(\{\theta^{-1}_{\overline{\Gamma},\mathrm{p}}\}(\{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)]))[F(\langle\langle\rangle,\mathrm{q}\rangle)]) \\
=_2 & \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}F(\gamma_\Gamma\delta\gamma^{-1}_\Delta\langle\langle\rangle,\mathrm{q}\rangle)]) \\
=_3 & \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}\rho^{-1}_{[],\overline{\Gamma}}d^{-1}_\Gamma\langle\iota\mathrm{p},\mathrm{q}\rangle\gamma_{F\Gamma}F(\delta)\gamma^{-1}_{F\Delta}\langle\langle\rangle,\mathrm{q}\rangle d_\Delta\rho_{[],\overline{\Delta}}F(\langle\langle\rangle,\mathrm{q}\rangle)) \\
= & \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[d^{-1}_\Gamma\langle\iota\mathrm{p},\mathrm{q}\rangle\gamma_{F\Gamma}F(\delta)\gamma^{-1}_{F\Delta}\langle\langle\rangle,\mathrm{q}\rangle d_\Delta\rho_{[],\overline{\Delta}}F(\langle\langle\rangle,\mathrm{q}\rangle))
\end{array}$$

where (1) is by the previous calculation, (2) is by composition of coercions and (3) is by preservation of democracy by $F$.

We now focus on the subterm $d_\Delta\rho_{[],\overline{\Delta}}F(\langle\langle\rangle,\mathrm{q}\rangle)$:

$$\begin{array}{rll}
d_\Delta\rho_{[],\overline{\Delta}}F(\langle\langle\rangle,\mathrm{q}\rangle) & =_1 & d_\Delta\rho_{[],\overline{\Delta}}\rho^{-1}_{[],\overline{\Delta}}\langle F(\langle\rangle),\{\theta^{-1}_{\overline{\Delta},\langle\rangle}\}(\sigma^{\overline{\Delta}[\langle\rangle]}_{\Gamma\overline{\Delta}[\langle\rangle]}(q))\rangle \\
& =_2 & d_\Delta\langle F\langle\rangle,\{\theta^{-1}_{\overline{\Delta},\langle\rangle}\}(\{\theta_{\overline{\Delta}[\langle\rangle],\mathrm{p}}\}(\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]))\rangle \\
& =_3 & d_\Delta\langle\iota\langle\rangle,\{\theta^{-1}_{\overline{\Delta},\langle\rangle}\theta_{\overline{\Delta}[\langle\rangle],\mathrm{p}}\}(\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}])\rangle \\
& =_4 & d_\Delta\langle\iota\langle\rangle,\{\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{\overline{\Delta},\langle\rangle})\}(\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}])\rangle
\end{array}$$

where (1) is by preservation of substitution extension (Proposition 4), (2) is by equality (b) for preservation of the second projection, (3) is by preservation of the terminal object and composition of coercions, (4) is by coherence of $\theta$.

We now focus on the subterm $\{\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{\overline{\Delta},\langle\rangle})\}(\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}])$:

$$\begin{array}{rll}
\{\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{\overline{\Delta},\langle\rangle})\}(\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]) & =_1 & \mathrm{q}[\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{\overline{\Delta},\langle\rangle})\langle\mathrm{id},\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]\rangle] \\
& =_2 & \mathrm{q}[\langle\mathrm{p},\mathrm{q}[\theta^{-1}_{\overline{\Delta},\langle\rangle}\langle(F\mathrm{p})\mathrm{p},\mathrm{q}\rangle]\rangle\langle\mathrm{id},\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]\rangle] \\
& = & \mathrm{q}[\theta^{-1}_{\overline{\Delta},\langle\rangle}\langle(F\mathrm{p}),\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]\rangle] \\
& =_3 & \mathrm{q}[\theta^{-1}_{\overline{\Delta},\langle\rangle}\langle\mathrm{p}\rho_{\Gamma,\overline{\Delta}[\langle\rangle]},\mathrm{q}[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]\rangle \\
& = & \mathrm{q}[\theta^{-1}_{\overline{\Delta},\langle\rangle}\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]
\end{array}$$

where (1) is by definition of coercions, (2) is by definition of $\boldsymbol{T'}$, (3) is by equality (a) for preservation of the first projection, and the rest is by basic manipulations of cwf combinators.

Replacing that in the previous calculation, we further calculate:

$$
\begin{aligned}
d_\Delta \rho_{[],\overline{\Delta}} F(\langle\langle\rangle, \mathrm{q}\rangle) &= d_\Delta \langle \iota\langle\rangle, \mathrm{q}[\theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]\rangle \\
&=_1 d_\Delta \langle \iota\langle\rangle, \mathrm{q}\rangle \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]} \\
&= \langle \mathrm{p} d_\Delta \langle \iota\langle\rangle, \mathrm{q}\rangle, \mathrm{q}[d_\Delta \langle \iota\langle\rangle, \mathrm{q}\rangle]\rangle \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]} \\
&=_1 \langle \mathrm{p}, \mathrm{q}[d_\Delta \langle \iota\langle\rangle, \mathrm{q}\rangle]\rangle \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]} \\
&=_2 \boldsymbol{T'}(\iota\langle\rangle)(d_\Delta) \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]}
\end{aligned}
$$

where (1) is by terminality of [], (2) is by definition of $\boldsymbol{T'}$, and the rest is by basic manipulations of cwf combinators.

Using this, we can finish converting $\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle])$:

$$
\begin{aligned}
\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) &= \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[d_\Gamma^{-1}\langle \iota\mathrm{p}, \mathrm{q}\rangle \gamma_{F\Gamma} F(\delta) \gamma^{-1}_{F\Delta} \langle\langle\rangle, \mathrm{q}\rangle d_\Delta \rho_{[],\overline{\Delta}} F(\langle\langle\rangle, \mathrm{q}\rangle)) \\
&=_1 \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[d_\Gamma^{-1}\langle \iota\mathrm{p}, \mathrm{q}\rangle \gamma_{F\Gamma} F\delta \gamma^{-1}_{F\Delta} \langle\langle\rangle, \mathrm{q}\rangle \boldsymbol{T'}(\iota\langle\rangle)(d_\Delta) \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]) \\
&=_2 \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[d_\Gamma^{-1}\langle \iota\langle\rangle, \mathrm{q}[\gamma_{F\Gamma} F\delta \gamma^{-1}_{F\Delta} \langle\langle\rangle, \mathrm{q}\rangle \boldsymbol{T'}(\iota\langle\rangle)(d_\Delta) \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]\rangle]) \\
&=_3 \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[d_\Gamma^{-1}\langle \iota\langle\rangle, \overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle][\boldsymbol{T'}(\iota\langle\rangle)(d_\Delta) \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]\rangle]) \\
&= \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[\langle \mathrm{p}, \mathrm{q}[d_\Gamma^{-1}\langle \iota\langle\rangle\mathrm{p}, \mathrm{q}\rangle]\rangle \langle \mathrm{id}, \overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle][\boldsymbol{T'}(\iota\langle\rangle)(d_\Delta) \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]\rangle]) \\
&=_4 \{\theta_{\overline{\Gamma},\langle\rangle}\}(\{\boldsymbol{T'}(\iota\langle\rangle)(d_\Gamma^{-1})\}(\overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle][\boldsymbol{T'}(\iota\langle\rangle)(d_\Delta) \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]}])) \\
&=_5 \{\theta_{\overline{\Gamma},\langle\rangle} \boldsymbol{T'}(\iota\langle\rangle)(d_\Gamma^{-1})\}(\overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle][\boldsymbol{T'}(\iota\langle\rangle)(d_\Delta) \theta^{-1}_{\overline{\Delta},\langle\rangle} \rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]) \\
&= C^{-1}(\overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle])
\end{aligned}
$$

where (1) is by the previous calculation, (2) is by terminality of [] and basic manipulation of cwf combinators, (3) is by definition of democracy (Definition 6), (4) is by defintion of $\boldsymbol{T'}$ and of coercions, (5) is by composition of coercions, and the rest is either by definition or basic manipulation of cwf combinators.

We get the required expression. The case of Equation (2) is similar but less intricate, so we skip the details. □

*Proof of Lemma 2.* Consider a chosen dependent product in $\mathbb{C}$. We want to characterize the fact that its image by $F$ is still a dependent product diagram, *i.e.* that the following is a dependent product.

$$
\begin{tikzcd}
& F(\Gamma\cdot A\cdot\Pi(A,B)[\mathrm{p}]) \arrow[r, "F(\langle \mathrm{pp},\mathrm{q}\rangle)"] \arrow[dl, bend right, "F(ev)"'] \arrow[d, "F(\mathrm{p})"] & F(\Gamma\cdot\Pi(A,B)) \arrow[d, "F(\mathrm{p})"] \\
F(\Gamma\cdot A\cdot B) \arrow[r, "F(\mathrm{p})"] & F(\Gamma\cdot A) \arrow[r, "F(\mathrm{p})"] & F(\Gamma)
\end{tikzcd}
$$

Let us denote this diagram by $\mathcal{D}$. By preservation of cwf structure, the "base" $F(\Gamma{\cdot}A{\cdot}B) \to$

$F(\Gamma\cdot A) \to F\Gamma$ of $\mathcal{D}$ is isomorphic to the following chain of projections:

$$\begin{tikzcd}
F(\Gamma\cdot A\cdot B) \arrow[r, "F(\mathrm{p})"] \arrow[d, "\langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[\rho_{\Gamma\cdot A,B}]\rangle"] & F(\Gamma\cdot A) \arrow[r, "F(\mathrm{p})"] \arrow[d, "\rho_{\Gamma\cdot A}"] & F\Gamma \arrow[d, "\mathrm{id}"] \\
F\Gamma\cdot\sigma_\Gamma(A)\cdot\sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}] \arrow[r, "\mathrm{p}"] & F\Gamma\cdot\sigma_\Gamma(A) \arrow[r, "\mathrm{p}"] & F\Gamma
\end{tikzcd}$$

The fact that $\mathcal{D}$ is a dependent product diagram is then equivalent to the existence of a (necessarily unique) isomorphism between it and the chosen dependent product diagram for $\Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])$ extending the isomorphism between their bases displayed earlier. In other word, $\mathcal{D}$ is a dependent product diagram iff there exists $i_{A,B} : \sigma_\Gamma(\Pi(A,B)) \to \Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma\cdot A}])$ and $f : F(\Gamma\cdot A\cdot\Pi(A,B)[\mathrm{p}]) \to F\Gamma\cdot\sigma_\Gamma(A)\cdot \Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])[\mathrm{p}]$, preserving the isomorphism between the bases of the two diagrams, and such that the following diagram commutes:

$$\begin{tikzcd}
F(\Gamma\cdot A\cdot B) \arrow[d, "\langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[\rho_{\Gamma\cdot A,B}]\rangle"] & F(\Gamma\cdot A\cdot\Pi(A,B)[\mathrm{p}]) \arrow[l, "F(ev)"'] \arrow[r, "F(\langle \mathrm{pp}, \mathrm{q}\rangle)"] \arrow[d, "f"] & F(\Gamma\cdot\Pi(A,B)) \arrow[d, "i_{A,B}\rho_{\Gamma,\Pi(A,B)}"] \\
F\Gamma\cdot\sigma_\Gamma(A)\cdot\sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}] & F\Gamma\cdot\sigma_\Gamma(A)\cdot\Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])[\mathrm{p}] \arrow[l, "ev"'] \arrow[r, "\langle \mathrm{pp}, \mathrm{q}\rangle"] & F\Gamma\cdot\Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])
\end{tikzcd}$$

It follows that necessarily, $f = \langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[i_{A,B}\rho_{\Gamma,\Pi(A,B)}F(\langle \mathrm{pp}, \mathrm{q}\rangle)]\rangle$, making the right hand side square commute.

We have established that $F$ preserves dependent products if and only if for any $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma\cdot A)$, the following equality holds:

$$ev\langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[i_{A,B}\rho_{\Gamma,\Pi(A,B)}F(\langle \mathrm{pp}, \mathrm{q}\rangle)]\rangle \quad = \quad \langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[\rho_{\Gamma\cdot A,B}]\rangle F(ev) \qquad (3)$$

Clearly, this holds if and only if for any $\delta : \Delta \to \Gamma$ in $\mathbb{C}$, for any $c : \Delta \vdash \Pi(A,B)[\delta]$ and $a : \Delta \vdash A[\delta]$, the equality holds when both sides are pre-composed by $F(\langle \delta, a, c\rangle)$. We evaluate the left hand side of Equation (3) pre-composed by $F(\langle \delta, a, c\rangle)$.

$$\begin{aligned}
& ev\langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[i_{A,B}\rho_{\Gamma,\Pi(A,B)}F(\langle \mathrm{pp}, \mathrm{q}\rangle)]\rangle F(\langle \delta, a, c\rangle) \\
=_1 \quad & \langle \mathrm{p}, \mathrm{ap}(\mathrm{q}, \mathrm{q}[\mathrm{p}])\rangle \langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[i_{A,B}\rho_{\Gamma,\Pi(A,B)}F(\langle \mathrm{pp}, \mathrm{q}\rangle)]\rangle F(\langle \delta, a, c\rangle) \\
= \quad & \langle \mathrm{p}, \mathrm{ap}(\mathrm{q}, \mathrm{q}[\mathrm{p}])\rangle \langle \rho_{\Gamma,A}F(\langle \delta, a\rangle), \mathrm{q}[i_{A,B}\rho_{\Gamma,\Pi(A,B)}F(\langle \delta, c\rangle)]\rangle \\
= \quad & \langle \rho_{\Gamma,A}F(\langle \delta, a\rangle), \mathrm{ap}(\mathrm{q}[i_{A,B}\rho_{\Gamma,\Pi(A,B)}F(\langle \delta, c\rangle)], \mathrm{q}[\rho_{\Gamma,A}F(\langle \delta, a\rangle)])\rangle \\
=_2 \quad & \langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_\Delta(a)), \mathrm{ap}(\mathrm{q}[i_{A,B}\langle F\delta, \{\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma^{\Pi(A,B)[\delta]}_\Delta(c))\rangle], \mathrm{q}[\langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_\Delta(a))\rangle])\rangle \\
= \quad & \langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_\Delta(a)), \mathrm{ap}(\mathrm{q}[i_{A,B}\langle F\delta, \{\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma^{\Pi(A,B)[\delta]}_\Delta(c))\rangle], \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_\Delta(a))\rangle
\end{aligned}$$

where (1) is by definition of $ev$, (2) is by preservation of substitution extension (Proposition 4), and the other equalities are by standard manipulation of cwf combinators. We further evaluate the subterm $\mathrm{q}[i_{A,B}\langle F\delta, \{\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma^{\Pi(A,B)[\delta]}_\Delta(c))\rangle]$:

$$\begin{aligned}
& \mathrm{q}[i_{A,B}\langle F\delta, \{\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma_\Delta^{\Pi(A,B)[\delta]}(c)\rangle] \\
= \quad & \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[i_{A,B}\langle (F\delta)\mathrm{p}, \mathrm{q}\rangle]\rangle\langle \mathrm{id}, \{\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma_\Delta^{\Pi(A,B)[\delta]}(c))\rangle] \\
=_1 \quad & \{\boldsymbol{T'}(F\delta)(i_{A,B})\}(\{\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma_\Delta^{\Pi(A,B)[\delta]}(c))) \\
=_2 \quad & \{\boldsymbol{T'}(F\delta)(i_{A,B})\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma_\Delta^{\Pi(A,B)[\delta]}(c))
\end{aligned}$$

where (1) is by definition of $\boldsymbol{T'}$ and of the application of coercions $\{\theta\}(a)$, and (2) is by compatibility of the application of coercions with composition. The other equalities are by standard manipulations of cwf combinators. So far, we have established that the left hand side of Equation (3) pre-composed with $F(\langle \delta, a, c\rangle)$ is equal to:

$$\langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma_\Delta^{A[\delta]}(a)), \mathrm{ap}(\{\boldsymbol{T'}(F\delta)(i_{A,B})\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma_\Delta^{\Pi(A,B)[\delta]}(c)), \{\theta^{-1}_{A,\delta}\}(\sigma_\Delta^{A[\delta]}(a)))\rangle$$

Let us now evaluate the right hand side. We calculate:

$$\begin{aligned}
& \langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[\rho_{\Gamma\cdot A,B}]\rangle F(ev)F(\langle\delta,a,c\rangle) \\
=_1 \quad & \langle \rho_{\Gamma,A}(F\mathrm{p}), \mathrm{q}[\rho_{\Gamma\cdot A,B}]\rangle F(\langle \mathrm{p}, \mathrm{ap}(\mathrm{q}, \mathrm{q}[\mathrm{p}])\rangle)F(\langle\delta,a,c\rangle) \\
= \quad & \langle \rho_{\Gamma,A}F(\langle\delta,a\rangle), \mathrm{q}[\rho_{\Gamma\cdot A,B}F(\langle \delta, a, \mathrm{ap}(c,a)\rangle)]\rangle \\
=_2 \quad & \langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma_\Delta^{A[\delta]}(a)), \mathrm{q}[\langle F(\langle\delta,a\rangle), \{\theta^{-1}_{B,\langle\delta,a\rangle}\}(\sigma_\Delta^{B[\langle\delta,a\rangle]}(\mathrm{ap}(c,a)))\rangle]\rangle \\
= \quad & \langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma_\Delta^{A[\delta]}(a)), \{\theta^{-1}_{B,\langle\delta,a\rangle}\}(\sigma_\Delta^{B[\langle\delta,a\rangle]}(\mathrm{ap}(c,a)))\rangle
\end{aligned}$$

where (1), is by definition of $ev$, (2) is by preservation of substitution extension (Proposition 4), and the other equalities are by standard manipulations of cwf combinators.

So we have established that Equation (3) pre-composed with $F(\langle\delta,a,c\rangle)$ amounts to the equality between

$$\langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma_\Delta^{A[\delta]}(a)), \mathrm{ap}(\{\boldsymbol{T'}(F\delta)(i_{A,B})\theta^{-1}_{\Pi(A,B),\delta}\}(\sigma_\Delta^{\Pi(A,B)[\delta]}(c)), \{\theta^{-1}_{A,\delta}\}(\sigma_\Delta^{A[\delta]}(a)))\rangle$$

and

$$\langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma_\Delta^{A[\delta]}(a)), \{\theta^{-1}_{B,\langle\delta,a\rangle}\}(\sigma_\Delta^{B[\langle\delta,a\rangle]}(\mathrm{ap}(c,a)))\rangle$$

whose right projection is exactly the condition required of pseudo cwf-morphisms for preservation of Π-types. □

## Appendix C. Additional proofs about the pseudofunctor $H$

*Proof of Lemma 5.* We set $\psi_{\Gamma,A} = \langle \mathrm{p}, \mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle : F\Gamma\cdot\sigma_\Gamma A \to F\Gamma\cdot\tau_\Gamma(A)[\phi_\Gamma]$. To check the coherence law, we apply the universal property of a well-chosen pullback square (exploiting the fact that $G$ preserves finite limits).

$$\begin{tikzcd}
F\Delta\cdot\tau_\Delta(A[\delta])[\phi_\Delta] \arrow[r, "\langle\phi_\Delta \mathrm{p},\mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"] \arrow[dr, phantom, "\lrcorner", very near start] & G\Delta\cdot\tau_\Delta(A[\delta]) \arrow[r, "\rho'_{\Gamma,A}G(\langle(\delta\mathrm{p},\mathrm{q}\rangle)(\rho'_{\Delta,A[\delta]})^{-1}"] \arrow[d, "\mathrm{p}"] \arrow[dr, phantom, "\lrcorner", very near start] & G\Gamma\cdot\tau_\Gamma(A) \arrow[d, "\mathrm{p}"] \\
F\Delta \arrow[r, "\phi_\Delta"'] & G\Delta \arrow[r, "G\delta"'] & G\Gamma
\end{tikzcd}$$

The two paths $\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A})$ and $\psi_{\Delta,A[\delta]}\theta_{A,\delta}$ of the coherence diagram behave in the same way with respect to this pullback. Here is the calculation for the first path of the coherence diagram:

$$
\begin{array}{ll}
& \rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle)(\rho'_{\Delta,A[\delta]})^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
=_1 & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
=_2 & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\langle\mathrm{p},\mathrm{q}[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\mathrm{p}\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle,\mathrm{q}[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}]\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
=_2 & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\langle\mathrm{p},\mathrm{q}[\psi_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
= & \langle(G\delta)\phi_\Delta\mathrm{p},\mathrm{q}[\psi_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
=_3 & \langle(G\delta)\phi_\Delta\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
=_4 & \langle\phi_\Gamma(F\delta)\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
= & \langle\phi_\Gamma\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle\langle(F\delta)\mathrm{p},\mathrm{q}\rangle \\
=_5 & \langle\mathrm{p}\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle\langle(F\delta)\mathrm{p},\mathrm{q}\rangle \\
= & \rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle
\end{array}
$$

where (1) is by Lemma 8, (2) is by definition of $\boldsymbol{T'}$, (3) is by definition of $\psi$ and a basic manipulation of cwf combinators, (4) is by naturality of $\phi$, (5) is by naturality of $\phi$ again and law (a) for preservation of the first projection, and the rest is by basic manipulation of cwf combinators.

We now simplify the second path in the diagram:

$$
\begin{array}{ll}
& \rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle){\rho'_{\Delta,A[\delta]}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\psi_{\Delta,A[\delta]}\theta_{A,\delta} \\
=_1 & \rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle){\rho'_{\Delta,A[\delta]}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\langle\mathrm{p},\mathrm{q}[\rho'_{\Delta,A[\delta]}\phi_{\Delta\cdot A[\delta]}\rho^{-1}_{\Delta,A[\delta]}]\rangle\theta_{A,\delta} \\
= & \rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle){\rho'_{\Delta,A[\delta]}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}[\rho'_{\Delta,A[\delta]}\phi_{\Delta\cdot A[\delta]}\rho^{-1}_{\Delta,A[\delta]}]\rangle\theta_{A,\delta} \\
=_2 & \rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle){\rho'_{\Delta,A[\delta]}}^{-1}\langle\mathrm{p}\rho'_{\Delta,A[\delta]}\phi_{\Delta\cdot A[\delta]}\rho^{-1}_{\Delta,A[\delta]},\mathrm{q}[\rho'_{\Delta,A[\delta]}\phi_{\Delta\cdot A[\delta]}\rho^{-1}_{\Delta,A[\delta]}]\rangle\theta_{A,\delta} \\
= & \rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle){\rho'_{\Delta,A[\delta]}}^{-1}\rho'_{\Delta,A[\delta]}\phi_{\Delta\cdot A[\delta]}\rho^{-1}_{\Delta,A[\delta]}\theta_{A,\delta} \\
= & \rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle)\phi_{\Delta\cdot A[\delta]}\rho^{-1}_{\Delta,A[\delta]}\theta_{A,\delta} \\
=_3 & \rho'_{\Gamma,A}\phi_{\Gamma\cdot A}F(\langle\delta\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}\theta_{A,\delta} \\
= & \rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\rho_{\Gamma,A}F(\langle\delta\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}\theta_{A,\delta} \\
=_4 & \rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle
\end{array}
$$

where (1) is by definition of $\psi$, (2) is by naturality of $\phi$ and equality (a) for the preservation of the first projection, (3) is by naturality of $\phi$, (4) is Lemma 8 and the rest are basic manipulations of cwf combinators. □

*Proof of Lemma 6.* The first equality is a straightforward verification. The second requires a more involved calculation similar to the one used to prove Lemma 5. Assume that we have the following situation:

$$(\mathbb{C}, T) \underset{(G,\tau)}{\overset{(F,\sigma)}{\rightrightarrows}} (\mathbb{C}', T') \underset{(G',\tau')}{\overset{(F',\sigma')}{\rightrightarrows}} (\mathbb{C}'', T'')$$

Let us call $\theta$ and $\rho$ the components of $(F, \sigma)$, $\theta'$ and $\rho'$ the components of $(F', \sigma')$, $\vartheta$ and $\varrho$ the components of $(G, \tau)$ and $\vartheta'$ and $\varrho'$ the components of $(G', \tau')$. Let us also consider natural transformations $\phi : F \stackrel{\bullet}{\to} G$ and $\phi' : F' \stackrel{\bullet}{\to} G'$. Let us recall that the vertical composition of pseudo cwf-transformations follows those of 2-cells in the 2-category of indexed categories over arbitrary bases, which means $(\phi, \psi_\phi)(\phi', \psi_{\phi'}) = (\phi\phi', m)$, where $m_{\Gamma,A}$ is obtained by:

$$\sigma'_{F\Gamma}(\sigma_\Gamma A) \xrightarrow{\boldsymbol{\sigma'}_{F\Gamma}((\psi_\phi)_{\Gamma,A})} \sigma'_{F\Gamma}(\tau_\Gamma A[\phi_\Gamma]) \xrightarrow{{\theta'_{\tau_\Gamma A,\phi_\Gamma}}^{-1}} \sigma'_{G\Gamma}(\tau_\Gamma A)[F'\phi_\Gamma] \xrightarrow{\boldsymbol{T''}(F'\phi_\Gamma)((\psi_{\phi'})_{G\Gamma,\tau_\Gamma A})} \tau'_{G\Gamma}(\tau_\Gamma A)[\phi'_{G\Gamma}F'(\phi_\Gamma)]$$

which the following calculation relates to $(\psi_{\phi\phi'})_{\Gamma,A}$:

$$\begin{array}{lll}
m_{\Gamma,A} & =_1 & \boldsymbol{T''}(F'\phi_\Gamma)((\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}){\theta'_{\tau_\Gamma A,\phi_\Gamma}}^{-1}\boldsymbol{\sigma'}_{F\Gamma}((\psi_\phi)_{\Gamma,A}) \\
& =_2 & \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\langle (F'\phi_\Gamma)\mathrm{p}, \mathrm{q}\rangle]\rangle {\theta'_{\tau_\Gamma A,\phi_\Gamma}}^{-1} \rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]} F'((\psi_\phi)_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1} \\
& =_3 & \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\langle (F'\phi_\Gamma)\mathrm{p}, \mathrm{q}\rangle {\theta'_{\tau_\Gamma A,\phi_\Gamma}}^{-1} \rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]} F'((\psi_\phi)_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& =_4 & \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle){\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}}^{-1}\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}F'((\psi_\phi)_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& = & \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& =_5 & \langle \mathrm{p}, \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[\varrho'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}{\rho'_{G\Gamma,\tau_\Gamma A}}^{-1}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle]\rangle \\
& = & \langle \mathrm{p}, \mathrm{q}[\varrho'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& =_6 & \langle \mathrm{p}, \mathrm{q}[\varrho'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle\langle \mathrm{p}, \mathrm{q}[\varrho_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& = & \langle \mathrm{p}, \mathrm{q}[\varrho'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}[\varrho_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& =_7 & \langle \mathrm{p}, \mathrm{q}[\varrho'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\langle \mathrm{p}\varrho_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}, \mathrm{q}[\varrho_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& = & \langle \mathrm{p}, \mathrm{q}[\varrho'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\varrho_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& =_8 & \langle \mathrm{p}, \mathrm{q}[\varrho'_{G\Gamma,\tau_\Gamma A}G'(\varrho_{\Gamma,A})\phi'_{G(\Gamma\cdot A)}F'(\phi_{\Gamma\cdot A})F'(\rho^{-1}_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1}]\rangle \\
& =_9 & \langle \mathrm{p}, \mathrm{q}[\rho^{G'G}_{\Gamma,A}(\phi\phi')_{\Gamma\cdot A}{\rho^{F'F}_{\Gamma,A}}^{-1}]\rangle \\
& =_{10} & (\psi_{\phi\phi'})_{\Gamma,A}
\end{array}$$

where (1) is by definition of $m$, (2) is by definition of $\boldsymbol{T''}$ and $\boldsymbol{\sigma'}$. The step (3) exploits basic manipulation of cwf combinators along with the equality

$$\mathrm{p}{\theta'_{\tau_\Gamma A,\phi_\Gamma}}^{-1}\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}F'((\psi_\phi)_{\Gamma,A}){\rho'_{F\Gamma,\sigma_\Gamma A}}^{-1} = \mathrm{p}$$

which itself follows from the fact that each instance of $\theta$ and $\psi$ commutes with the first projection, and from the Equality (a) for the preservation of the first projection. Step (4) is by Lemma 8, step (5) is by definition of $\psi_{\phi'}$, (6) by definition of $\psi_\phi$, (7) is by naturality of $\phi'$ and equality (a) for preservation of the first projection, (8) is by naturality of $\phi'$,

(9) is by definition of $\rho^{G'G}$, of $\phi\phi'$ and $\rho^{F'F}$, and (10) is by definition of $\psi_{\phi\phi'}$. The rest is by basic manipulation of cwf combinators.

Finally, the third equality follows from the remark that by definition of $\psi_{1_F}$, we have $(\psi_{1_F})_{\Gamma,A} = \mathrm{id}_{\Gamma\sigma_\Gamma A}$ for all $\Gamma$, $A$. $\square$

**Appendix. List of Notations**